

On coverage and local radial rates of DDM-credible sets

Eduard Belitser
VU University Amsterdam

For a general statistical model, we introduce the notion of *data dependent measure* (DDM) on the model parameter. Typical examples of DDM are the posterior distributions. Like for posteriors, the quality of a DDM is characterized by the contraction rate which we allow to be local, i.e., depending on the parameter. We construct confidence sets as *DDM-credible sets* and address the issue of optimality of such sets, via a trade-off between its “size” (the *local radial rate*) and its coverage probability. In the mildly ill-posed inverse signal-in-white-noise model, we construct a DDM as empirical Bayes posterior with respect to a certain prior, and define its (default) credible set. Then we introduce *excessive bias restriction* (EBR), more general than *self-similarity* and *polished tail condition* recently studied in the literature. Under EBR, we establish the confidence optimality of our credible set with some local (*oracle*) radial rate. We also derive the oracle estimation inequality and the oracle DDM-contraction rate, non-asymptotically and uniformly in ℓ_2 . The obtained local results are more powerful than global: adaptive minimax results for a number of smoothness scales follow as consequence, in particular, the ones considered by Szabó et al. (2015) [21].

1 Introduction

Suppose we observe a random element $X^{(\varepsilon)} \sim P_0^{(\varepsilon)} \in \mathcal{P}^{(\varepsilon)}$, $X^{(\varepsilon)} \in \mathcal{X}^{(\varepsilon)}$ for some measurable space $(\mathcal{X}^{(\varepsilon)}, \mathcal{A}^{(\varepsilon)})$, where $\mathcal{A}^{(\varepsilon)}$ is a σ -algebra on $\mathcal{X}^{(\varepsilon)}$. In fact, we consider a sequence of observation models parametrized by $\varepsilon > 0$. Parameter ε is assumed to be known, it reflects in some sense the influx of information in the data $X^{(\varepsilon)}$ as $\varepsilon \rightarrow 0$. For instance, ε can be the variance of an additive noise, or $\varepsilon = n^{-1/2}$, where n is the sample size. To avoid overloaded notations, we will often drop the dependence on ε ; for example, $X = X^{(\varepsilon)}$ etc.

Let $\mathcal{P} = \{P_\theta, \theta \in \Theta\}$ and $P_0 = P_{\theta_0}$, where $\theta_0 \in \Theta \subseteq \mathcal{L}$ is an unknown parameter of interest belonging to some subset Θ of a linear space \mathcal{L} equipped with a (semi-)metric $d(\cdot, \cdot) : \mathcal{L} \times \mathcal{L} \rightarrow \mathbb{R}_+ = [0, +\infty)$. From now on, when we deal with probabilities of events in terms of the data $X \sim P_\theta$, we write P_θ . By θ_0 we denote the so called “true” value of the parameter θ to distinguish it from the variable $\theta \in \Theta$.

The aim is to construct an optimal (to be defined later) confidence set for the parameter $\theta_0 \in \Theta$ on the basis of observation $X \sim P_0 \in \mathcal{P}$, with a prescribed coverage probability. The convention throughout this paper is that we measure the size of a set by the smallest possible radius of a ball containing that set. It is thus sufficient to consider only confidence

MSC2010 subject classification: primary 62G15, 62C05; secondary 62G99.

Keywords and phrases: DDM-credible ball, excessive bias restriction, local radial rate.

balls as confidence sets. Let $B(l_0, r) = \{l \in \mathcal{L} : d(l_0, l) \leq r\}$ be the ball in space \mathcal{L} with center $l_0 \in \mathcal{L}$ and radius $r \geq 0$. Denote by $\mathcal{B}_{\mathcal{L}}$ the corresponding Borel σ -algebra on \mathcal{L} and by $\mathcal{B}_{\mathbb{R}}$ the usual Borel σ -algebra on \mathbb{R} . A general confidence ball for the parameter θ is of the form $B(\tilde{\theta}, \tilde{r}) = \{\theta \in \mathcal{L} : d(\theta, \tilde{\theta}) \leq \tilde{r}\}$, with some *data dependent center* (DD-center) $\tilde{\theta} = \tilde{\theta}(X) = \tilde{\theta}(X, \varepsilon)$, $\tilde{\theta} : \mathcal{X} \rightarrow \Theta$, and some *data dependent radius* (DD-radius) $\tilde{r} = \tilde{r}(X) = \tilde{r}(X, \varepsilon)$, $\tilde{r} : \mathcal{X} \rightarrow \mathbb{R}_+ = \{a \in \mathbb{R} : a \geq 0\}$. The quantities $\tilde{\theta}$ and \tilde{r} are $(\mathcal{A}, \mathcal{B}_{\mathbb{R}})$ -measurable functions of the data.

Suppose we are given a *data dependent measure* (DDM) $P(\cdot|X)$ on Θ (we will say: a DDM on the parameter θ). In order to settle the measurability issue for the rest of the paper, by DDM we will always mean a measurable probability measure in the sense that for all $x \in \mathcal{X}$ the quantity $P(\cdot|X = x)$ is a probability measure on $(\mathcal{B}_{\mathcal{L}}, \Theta)$ (can be relaxed to P_{θ_0} -almost all $x \in \mathcal{X}$, for all $\theta_0 \in \Theta$) and $P(B|X)$ is \mathcal{A} -measurable for each $B \in \mathcal{B}_{\mathcal{L}}$. Typically, a DDM is obtained by using a Bayesian approach, as the resulting posterior (or empirical Bayes posterior) distribution with respect to some prior on Θ , see Supplement for more details on how Bayes approach yields DDM's. We slightly abuse the traditional notation $P(\cdot|X)$ because in general a DDM does not have to be a conditional distribution. Notice that empirical Bayes posteriors are, strictly speaking, not conditional distributions either. Other examples considered in the literature that fall under the category of DDMs are (generalized) fiducial distributions and bootstrap.

A “good” DDM can be used for all kind of inference: e.g., estimation, construction of confidence sets (nowadays termed as *uncertainty quantification*), testing. As to confidence sets, given a DDM $P(\cdot|X)$ on Θ , we can take a *DDM-credible set* $C_{\alpha}(X)$ of level $\alpha \in [0, 1]$, i.e., $P(\theta \in C_{\alpha}(X)|X) \geq \alpha$, as a candidate confidence set. In this paper we focus on the following, for now loosely formulated, question:

When does DDM-credibility lead to confidence?

Let us specify the optimality framework for confidence sets. We would like to construct such a confidence ball $B(\hat{\theta}, C\hat{r})$ that for any $\alpha_1, \alpha_2 \in (0, 1]$ and some functional $r_{\varepsilon}(\theta) = r_{\varepsilon}(\theta)$, $r_{\varepsilon} : \Theta \rightarrow \mathbb{R}_+$, there exists $C, c > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ with some $\varepsilon_0 > 0$,

$$\sup_{\theta \in \Theta_{cov}} P_{\theta}(\theta \notin B(\hat{\theta}, C\hat{r})) \leq \alpha_1, \quad \sup_{\theta \in \Theta_{size}} P_{\theta}(\hat{r} \geq cr_{\varepsilon}(\theta)) \leq \alpha_2, \quad (1)$$

where $\Theta_{cov}, \Theta_{size} \subseteq \Theta$. In some papers, a confidence set satisfying the first relation in (1) is called *honest* over Θ_{cov} . The quantity $r_{\varepsilon}(\theta)$ has the meaning of the effective radius of the confidence ball $B(\hat{\theta}, C\hat{r})$. We call the quantity $r_{\varepsilon}(\theta)$ *radial rate*. Clearly, there are many possible radial rates, but it is desirable to find the “fastest” (i.e., smallest) radial rate $r_{\varepsilon}(\theta)$, for which the relations (1) hold for “massive” $\Theta_{cov}, \Theta_{size} \subseteq \Theta$, ideally for $\Theta_{cov} = \Theta_{size} = \Theta$. The two relations in (1) are called *coverage* and *size properties*. Asymptotic formulation is also possible: $\limsup_{\varepsilon \rightarrow 0}$ should be taken, constants α_1, α_2, C, c (possibly sets $\Theta_{cov}, \Theta_{size}$) can be allowed to depend on ε .

Thus the following optimality aspects are involved in the framework (1): the coverage, the radial rate, and the uniformity subsets $\Theta_{cov}, \Theta_{size}$. The optimality is basically a trade-off between these complementary aspects pushed to the utmost limits, when further improving upon one aspect leads to a deterioration in another aspect. For example, the

smaller the local radial rate $r_\varepsilon(\theta)$ in (1), the better. But if it is too small, the size requirement in (1) may hold uniformly only over some “thin” set $\Theta_{size} \subset \Theta$. On the other hand, if one insists on $\Theta_{cov} = \Theta_{size} = \Theta$, then it may be impossible to establish (1) for interesting (relatively small) radial rates $r_\varepsilon(\theta)$.

One approach to optimality is via minimax estimation framework. It is assumed that $\theta \in \Theta_\beta \subseteq \Theta$ for some “smoothness” parameter $\beta \in \mathcal{B}$, which may be known or unknown (non-adaptive or adaptive formulation). The key notion here is the so called *minimax rate* $R_\varepsilon(\Theta_\beta)$, see Supplement. The radial rate is taken to be $r_\varepsilon(\theta) = R_\varepsilon(\Theta_\beta)$, which is a global quantity as it is constant for all $\theta \in \Theta_\beta$. In the nonadaptive case, it can be shown that the minimax rate $R_\varepsilon(\Theta_\beta)$ is the *best global radial rate* (i.e., among all radial rates that are constant on Θ_β); see Supplement for more details.

An adaptation problem arises when, for a given family of models $\{\Theta_\beta, \beta \in \mathcal{B}\}$ (called *scale*), we only know that $\theta \in \Theta_\beta$ for some unknown $\beta \in \mathcal{B}$. In fact, $\theta \in \cup_{\beta \in \mathcal{B}} \Theta_\beta \subseteq \Theta$ and the problem becomes in general more difficult. For a $\Theta'_{cov} \subseteq \Theta$, we want to construct such a confidence ball $B(\hat{\theta}, C\hat{r})$ that

$$\sup_{\theta \in \Theta'_{cov}} P_\theta(\theta \notin B(\hat{\theta}, C\hat{r})) \leq \alpha_1, \quad \sup_{\theta \in \Theta_\beta} P_\theta(\hat{r} \geq cR_\varepsilon(\Theta_\beta)) \leq \alpha_2 \quad \forall \beta \in \mathcal{B}, \quad (2)$$

possibly in asymptotic setting: put $\limsup_{\varepsilon \rightarrow 0}$ in front of both sup in (2). Ideally, \mathcal{B} is “massive” and $\Theta'_{cov} \supseteq \Theta_\beta$. However, in general it is impossible to construct optimal (fully) adaptive confidence set in the minimax sense: the coverage requirement in (2) does not hold even for $\Theta'_{cov} = \Theta_\beta$. For the classical many normal means model, there are negative results in [15], [1], [7]; this is also discussed in [19]. A way to achieve adaptivity is to remove the so called *deceptive parameters* (in [21] they are called *inconvenient truths*) from Θ , i.e., consider a strictly smaller set $\Theta'_{cov} \subset \Theta$. Examples are: $\Theta'_{cov} = \Theta_{ss}$, the so called *self-similar* parameters (related to Sobolev/Besov scales) introduced in [18] and later studied in [5], [6], [21], [17], [20]; and $\Theta'_{cov} = \Theta_{pt}$, a more general class of *polished tail* parameters introduced in [21]. More literature on adaptive minimax confidence sets: [16], [3], [18], [13], [10], [11], [14], [5], [6], [17], [21], [22].

In all the above mentioned papers global minimax radial rates $R_\varepsilon(\Theta_\beta)$ (as in (2)) were studied. In this paper we allow local radial rates as in the framework (1). When applied appropriately, the local approach is actually more powerful and flexible. Namely, suppose that a local radial rate $r_\varepsilon(\theta)$ is such that, for some uniform $c > 0$,

$$r_\varepsilon(\theta) \leq cR_\varepsilon(\Theta_\beta), \quad \text{for all } \theta \in \Theta_\beta, \beta \in \mathcal{B}. \quad (3)$$

If in addition $\Theta'_{cov} \subseteq \Theta_{cov}$ and $\Theta_\beta \subseteq \Theta_{size}$ for all $\beta \in \mathcal{B}$, then the results of type (1) imply the results of type (2), *simultaneously for all scales* $\{\Theta_\beta, \beta \in \mathcal{B}\}$ for which (3) is satisfied. We say that the local radial rate $r_\varepsilon(\theta)$ *covers* these scales; more details are in Supplement.

In Section 2 we consider a general setting and present two types of conditions on a DDM $P(\cdot|X)$: the upper and lower bounds on the DDM-contraction rate in terms of a given local radial rate $r_\varepsilon(\theta_0)$. Roughly speaking, the upper bound condition means that the DDM $P(\cdot|X)$ contracts at θ_0 with the local rate at least $r_\varepsilon(\theta_0)$, from the P_{θ_0} -perspective; then one can also construct a DD-center $\hat{\theta}$ which is an estimator of θ_0 with the rate $r_\varepsilon(\theta_0)$.

The lower bound condition means that the DDM concentrates around the DD-center $\tilde{\theta}$ at a rate that is not faster than $r_\varepsilon(\theta_0)$. We show that the upper bound condition allows to control the size of the $P(\cdot|X)$ -credible ball, whereas the lower bound is in some sense the minimal condition for providing its sufficient P_{θ_0} -coverage.

In Section 3 we consider the mildly ill-posed inverse signal-in-white-noise model and implement the general approach of Section 2. We construct a DDM $P(\cdot|X)$, which is in fact the empirical Bayes posterior resulting from a certain two-level hierarchical prior. For the proposed DDM, we first prove the upper bound type result. Namely, we establish that the DDM $P(\cdot|X)$ contracts, from the P_{θ_0} -perspective, to θ_0 with the local rate $r_\varepsilon(\theta_0)$, which is the best (fastest) contraction rate over some family of DDMs (therefore also called *oracle rate*). The DDM contraction result is non-asymptotic and uniform in $\theta_0 \in \ell_2$. The local radial rate $r_\varepsilon(\theta_0)$ satisfies (3) for typical smoothness scales such as Sobolev and analytic ellipsoids, Sobolev hyperrectangles, tail classes, certain scales of Besov classes and ℓ_p -bodies. This means that we obtained, as consequence of our local result, the adaptive minimax contraction rate results over all these scales for the DDM $P(\cdot|X)$. An accompanying result is that, by using the DDM $P(\cdot|X)$, a DD-center $\tilde{\theta}$ can be constructed that converges to θ_0 also with the local rate $r_\varepsilon(\theta_0)$, thus also yielding the panorama of the minimax adaptive estimation results over all these scales simultaneously.

Although the upper bound results are of interest on its own, our main purpose is to construct an optimal (according to the framework (1)) confidence set. To this end, the established upper bound results imply the size relation for a $P(\cdot|X)$ -credible ball in (1) with the local radial rate $r_\varepsilon(\theta_0)$, uniformly over $\Theta_{size} = \ell_2$. For the coverage relation in (1) to hold, we also need the lower bound results. It turns out that the lower bound result can be established uniformly only over some $\Theta_{cov} \subset \ell_2$, which forms an actual restriction. This is in accordance with the above mentioned fact that it is impossible to construct optimal (fully) adaptive confidence set in the minimax sense. We propose a set $\Theta_{cov} = \Theta_{eb}$ of (non-deceptive) parameters satisfying the so called *excessive bias restriction* and derive the lower bound uniformly over this set. Combining the obtained upper and lower bounds, we establish the optimality (1) of a (default) DDM-credible ball with $\Theta_{cov} = \Theta_{eb}$, $\Theta_{size} = \ell_2$ and the local radial rate $r_\varepsilon(\theta_0)$. The class Θ_{eb} is more general than the earlier mentioned self-similar and polished tail parameters, namely, $\Theta_{ss} \subseteq \Theta_{pt} \subseteq \Theta_{eb}$. Moreover, the established (local) optimality (1) implies the global optimality (2) in the sense of adaptive minimaxity over all scales for which (3) is fulfilled, in particular for the ones considered by Szabó et al. (2015). In this paper, we primarily interested in non-asymptotic assertions, asymptotic versions can be readily obtained. Section 4 contains the proofs of the main results. The elaboration on some points and some background information related to the paper are provided in Supplement.

2 General DDM-based construction of confidence ball

2.1 DDM-credible ball

Suppose we are given a DDM $P(\cdot|X)$ on θ . The goal of this section is to construct a confidence set by using this DDM and to elaborate on its coverage and size. Recall that

our optimality framework is (1), with a local radial rate $r(\theta_0) = r_\varepsilon(\theta_0)$. In this section we are not concerned with specific choices for radial rates and simply suppose that we are given some local radial rate $r(\theta_0)$. As $X = X^{(\varepsilon)}$, the DDM $P(\cdot|X)$ depends on ε . Hence, so do all the DDM-based quantities. In this section we omit this dependence completely to ease the notations. The convention for the rest of this section is that all assumptions and claims hold for all $\varepsilon \in [0, \varepsilon_0]$ with some $\varepsilon_0 > 0$.

First we present the general construction of a confidence ball by using the DDM $P(\theta|X)$ and a DD-center $\hat{\theta} = \hat{\theta}(X)$. For a $\kappa \in (0, 1)$, define the DD-radius

$$\hat{r}_\kappa = \hat{r}(\kappa, X, \hat{\theta}) = \inf \{r : P(d(\theta, \hat{\theta}) \leq r|X) \geq 1 - \kappa\} \quad (4)$$

and then, for an $M > 0$, construct the confidence ball

$$B(\hat{\theta}, M\hat{r}_\kappa) = \{\theta \in \Theta : d(\theta, \hat{\theta}) \leq M\hat{r}_\kappa\}. \quad (5)$$

For $M = 1$, (5) is the smallest DDM-credible ball around $\hat{\theta}$ of level $1 - \kappa$. For a good DDM that concentrates around θ_0 from the P_{θ_0} -perspective (i.e., under $X \sim P_{\theta_0}$), a DDM-credible set should also be a good confidence set, but its P_{θ_0} -coverage is in general lower than $1 - \kappa$ because of uncertainty in the data. The multiplicative factor M , not dependent on ε , is intended to inflate the DDM-credible ball of level $1 - \kappa$ to account for this uncertainty.

Now we construct a confidence ball by using only the given DDM $P(\cdot|X)$, without a predetermined DD-center. For a $p \in (1/2, 1)$, define first

$$\hat{r}^* = \hat{r}^*(p) = \inf \{r : P(d(\theta, \theta') \leq r|X) \geq p \text{ for some } \theta' \in \Theta\}. \quad (6)$$

This is the smallest possible radius of DDM-credible ball of level p . Next, for some $\varsigma > 0$, take any (measurable function of data X) $\check{\theta} \in \Theta$ that satisfies

$$P(\theta : d(\theta, \check{\theta}) \leq (1 + \varsigma)\hat{r}^*|X) \geq p. \quad (7)$$

We call the constructed $\check{\theta} = \check{\theta}(p, \varsigma)$ *default DD-center*, with respect to the DDM $P(\cdot|X)$. In words, $\check{\theta} = \check{\theta}(p, \varsigma)$ is the center of the ball of nearly the smallest radius subject to the constraint that its DDM $P(\cdot|X)$ -mass is at least p .

Finally, define the *default DDM-credible ball*: for a $\kappa \in (0, 1)$,

$$\tilde{B} = \tilde{B}_M = \tilde{B}_{M, \kappa} = B(\check{\theta}, M\hat{r}_\kappa), \quad (8)$$

where $B(\check{\theta}, M\hat{r}_\kappa)$ is defined by (4) and (5), and $\check{\theta}$ is defined by (6) and (7).

2.2 Conditions

Here we present some conditions used later for establishing general statements about the coverage and the size of the confidence ball (5) (and (8)). For $\theta_0 \in \Theta$, $M, \delta \geq 0$, some local radial rate $r(\theta_0)$, some DDM $P(\cdot|X)$ and DD-center $\hat{\theta} = \hat{\theta}(X)$, introduce the following conditions.

(A1) For some $\phi_1(M) = \phi_1(M, \varepsilon, \theta_0, \hat{\theta}) \geq 0$, such that $\phi_1(M) \downarrow 0$ as $M \uparrow \infty$,

$$\mathbb{E}_{\theta_0}[\mathbb{P}(d(\theta, \hat{\theta}) \geq Mr(\theta_0)|X)] \leq \phi_1(M).$$

(A2) For some $\psi(\delta) = \psi(\delta, \varepsilon, \theta_0, \hat{\theta}) \geq 0$ such that $\psi(\delta) \downarrow 0$ as $\delta \downarrow 0$,

$$\mathbb{E}_{\theta_0}[\mathbb{P}(d(\theta, \hat{\theta}) \leq \delta r(\theta_0)|X)] \leq \psi(\delta).$$

(A3) For some $\phi_2(M) = \phi_2(M, \varepsilon, \theta_0, \hat{\theta}) \geq 0$ such that $\phi_2(M) \downarrow 0$ as $M \uparrow \infty$,

$$\mathbb{P}_{\theta_0}(d(\theta_0, \hat{\theta}) \geq Mr(\theta_0)) \leq \phi_2(M).$$

Conditions (A1)–(A3) trivially hold for the functions $\phi_1(M, \varepsilon, \theta_0, \hat{\theta}) = \mathbb{E}_{\theta_0}[\mathbb{P}(d(\theta, \hat{\theta}) \geq Mr(\theta_0)|X)]$, $\psi(\delta, \varepsilon, \theta_0, \hat{\theta}) = \mathbb{E}_{\theta_0}[\mathbb{P}(d(\theta, \hat{\theta}) \leq \delta r(\theta_0)|X)]$, $\phi_2(M, \varepsilon, \theta_0, \hat{\theta}) = \mathbb{P}_{\theta_0}(d(\theta_0, \hat{\theta}) \geq Mr(\theta_0))$. Conditions (A1)–(A3) become really useful when the functions ϕ_1, ψ, ϕ_2 do not depend on $\varepsilon \in (0, \varepsilon_0]$ and $\theta_0 \in \Theta_0$, for some $\varepsilon_0 > 0$ and $\Theta_0 \subseteq \Theta$ (preferably $\Theta_0 = \Theta$). Then (A1) means that $\mathbb{P}(\cdot|X)$ concentrates, from the \mathbb{P}_{θ_0} -perspective, around $\hat{\theta}$ with the radial rate at least $r(\theta_0)$, (A2) means that $\mathbb{P}(\cdot|X)$ concentrates around $\hat{\theta}$ with the radial rate at most $r(\theta_0)$. Condition (A3) means that the DD-center $\hat{\theta}$ is an estimator of θ_0 with the rate $r(\theta_0)$. Together (A1) and (A2) imply that $\mathbb{P}(\cdot|X)$ concentrates, from the \mathbb{P}_{θ_0} -perspective, on the spherical shell $\{\theta : \delta r(\theta_0) \leq d(\theta, \hat{\theta}) \leq Mr(\theta_0)\}$ for sufficiently small δ and large M .

Condition (A1) is reminiscent of the definition of the so called (global) posterior contraction rate $R_\varepsilon(\Theta)$ from the nonparametric Bayes literature: $\Pi(d(\theta_0, \theta) \geq MR_\varepsilon(\Theta)|X)$ should be small for sufficiently large M from the \mathbb{P}_{θ_0} -probability perspective. The following introduces a counterpart of a local contraction rate for a general DDM $\mathbb{P}(\cdot|X)$.

($\tilde{A}1$) For some $\varphi(M) = \varphi(M, \varepsilon, \theta_0) \geq 0$ such that $\varphi(M) \downarrow 0$ as $M \uparrow \infty$,

$$\mathbb{E}_{\theta_0}[\mathbb{P}(d(\theta_0, \theta) \geq Mr(\theta_0)|X)] \leq \varphi(M).$$

Clearly, condition (A1) is implied by conditions ($\tilde{A}1$) and (A3) for the function $\phi_1(M) = \phi_2(aM) + \varphi((1-a)M)$ with any $a \in (0, 1)$.

Introduce a strengthened version of condition (A2).

($\tilde{A}2$) For some $\psi(\delta) = \psi(\delta, \varepsilon, \theta_0) \geq 0$ such that $\psi(\delta) \downarrow 0$ as $\delta \downarrow 0$ and any DD-center $\tilde{\theta} = \tilde{\theta}(X)$, $\mathbb{E}_{\theta_0}[\mathbb{P}(d(\theta, \tilde{\theta}) \leq \delta r(\theta_0)|X)] \leq \psi(\delta)$.

The difference between ψ from (A2) and ψ from ($\tilde{A}2$) is that the latter does not depend on the DD-center. We keep however the same notation for the function ψ in ($\tilde{A}2$) as in (A2) without confusion as we are never going to use both conditions simultaneously.

Instead of non-asymptotic conditions, even in (regular) parametric models one typically verifies asymptotic versions. In Supplement we introduce asymptotic (as $\varepsilon \rightarrow 0$) versions of conditions (A1)–(A3), ($\tilde{A}1$)–($\tilde{A}2$) denoted as (AA1)–(AA3) and (A $\tilde{A}1$)–(A $\tilde{A}2$). The asymptotic versions of all the assertions below can be reproduced by using (AA1)–(AA3) instead of (A1)–(A3). More remarks about the conditions are in Supplement.

2.3 Conditions for default confidence ball

The following proposition claims that condition $(\tilde{A}1)$ implies conditions (A1) and (A3) for the default DD-center $\check{\theta}$ defined by (6)–(7), with appropriate choices of ϕ_1 and ϕ_2 . Hence, $(\tilde{A}1)$ – $(\tilde{A}2)$ imply $(\tilde{A}1)$ and (A2) which in turn imply (A1)–(A3) for $\check{\theta}$.

Proposition 1. *Let condition $(\tilde{A}1)$ be fulfilled with function $\varphi(M)$ and let the default DD-center $\check{\theta}$ be defined by (6) and (7). Then condition (A1) holds with function $\phi_1(M) = \varphi(aM/(2+\varsigma))/p + \varphi((1-a)M)$ for any $a \in (0, 1)$, and condition (A3) holds with function $\phi_2(M) = \varphi(M/(2+\varsigma))/p$.*

Proof. If (A3) holds true with $\phi_2(M) = \varphi(M/(2+\varsigma))/p$, then, by using this and $(\tilde{A}1)$, we obtain that, for any $a \in (0, 1)$,

$$\begin{aligned} \mathbb{E}_{\theta_0} [\mathbb{P}(d(\theta, \check{\theta}) \geq Mr(\theta_0)|X)] &\leq \mathbb{E}_{\theta_0} [\mathbb{P}(d(\theta, \theta_0) \geq aMr(\theta_0)|X)] \\ &\quad + \mathbb{E}_{\theta_0} [\mathbb{P}(d(\theta_0, \check{\theta}) \geq (1-a)Mr(\theta_0)|X)], \end{aligned}$$

which implies (A1) with $\phi_1(M) = \varphi(aM/(2+\varsigma))/p + \varphi((1-a)M)$.

Therefore, it remains to show (A3) with the function $\phi_2(M) = \varphi(M/(2+\varsigma))/p$. From $(\tilde{A}1)$ it follows by the Markov inequality that

$$\mathbb{P}_{\theta_0}(\mathbb{P}(\theta \in B(\theta_0, Mr(\theta_0)|X) \geq p) \geq 1 - \frac{\varphi(M)}{p}.$$

By (7), the ball $B(\check{\theta}, (1+\varsigma)\hat{r}^*)$ has $\mathbb{P}(\cdot|X)$ -probability at least p . If the ball $B(\theta_0, Mr(\theta_0))$ also has $\mathbb{P}(\cdot|X)$ -probability at least p (which happens with \mathbb{P}_{θ_0} -probability at least $1 - \frac{\varphi(M)}{p}$), then, firstly, $\hat{r}^* \leq Mr(\theta_0)$ by virtue of the definition (6) of \hat{r}^* , and, secondly, the balls $B(\check{\theta}, (1+\varsigma)\hat{r}^*)$ and $B(\theta_0, Mr(\theta_0))$ must intersect, otherwise the total $\mathbb{P}(\cdot|X)$ -mass would exceed $2p > 1$. Hence, by the triangle inequality, $d(\theta_0, \check{\theta}) \leq (1+\varsigma)\hat{r}^* + Mr(\theta_0) \leq (2+\varsigma)Mr(\theta_0)$, with \mathbb{P}_{θ_0} -probability at least $1 - \varphi(M)/p$. Hence, condition (A3) holds with $\phi_2(M) = \varphi(M/(2+\varsigma))/p$ for the default DD-center $\check{\theta}$. \square

Remark 1. Of course, \hat{r}^* depends on p and $\check{\theta}$ depends on both p and ς . We however skip this dependences from the notations by assuming from now on that $p = 2/3$ and $\varsigma = 1/2$. We take $a = 1/2$ in Proposition 1. According to Proposition 1, if condition $(\tilde{A}1)$ is fulfilled with function $\varphi(M)$, then conditions (A1) and (A3) hold for the default DD-center $\check{\theta}$, with the functions $\phi_1(M) = 3\varphi(M/5)/2 + \varphi(M/2)$ and $\phi_2(M) = 3\varphi(2M/5)/2$ respectively.

2.4 Coverage and size of the DDM-credible set

Recall that our main goal is to construct a confidence ball satisfying the optimality framework (1). In this subsection we present some simple general (coverage and size) properties of the DDM-credible ball $B(\hat{\theta}, M\hat{r}_\kappa)$ defined by (5) with a DDM $\mathbb{P}(\cdot|X)$ and a DD-center $\hat{\theta}$ satisfying (A1)–(A3). Next we briefly outline how these properties can be used to establish the optimality framework (1) in concrete settings.

The following proposition gives an upper bound for the coverage probability of the confidence ball (5).

Proposition 2. For a $\theta_0 \in \Theta$ and some radial rate $r(\theta_0)$, let $\kappa \in (0, 1)$ and the ball $B(\hat{\theta}, M\hat{r}_\kappa)$ be defined by (5) with a DDM $P(\cdot|X)$ and a DD-center $\hat{\theta}$ satisfying conditions (A2) and (A3). Then for any $M, \delta > 0$,

$$P_{\theta_0}(\theta_0 \notin B(\hat{\theta}, M\hat{r}_\kappa)) = P_{\theta_0}(d(\theta_0, \hat{\theta}) > M\hat{r}_\kappa) \leq \phi_2(M\delta) + \frac{\psi(\delta)}{1 - \kappa}.$$

Proof. By the Markov inequality, (4) and conditions (A2) and (A3), we derive

$$\begin{aligned} P_{\theta_0}(d(\theta_0, \hat{\theta}) > M\hat{r}_\kappa) &\leq P_{\theta_0}(d(\theta_0, \hat{\theta}) > M\hat{r}_\kappa, \hat{r}_\kappa \geq \delta r(\theta_0)) + P_{\theta_0}(\hat{r}_\kappa < \delta r(\theta_0)) \\ &\leq P_{\theta_0}(d(\theta_0, \hat{\theta}) > M\delta r(\theta_0)) + P_{\theta_0}(P(d(\theta, \hat{\theta}) \leq \delta r(\theta_0)|X) \geq 1 - \kappa) \\ &\leq \phi_2(M\delta) + \frac{E_{\theta_0}(P(d(\theta, \hat{\theta}) \leq \delta r(\theta_0)|X))}{1 - \kappa} \leq \phi_2(M\delta) + \frac{\psi(\delta)}{1 - \kappa}. \end{aligned} \quad \square$$

It is not difficult to see that (A2) guarantees that the rate $r(\theta_0)$ is actually sharp. Indeed, as is already derived in the proof of Proposition 2,

$$P_{\theta_0}(\hat{r}_\kappa \leq \delta r(\theta_0)) \leq \frac{\psi(\delta)}{1 - \kappa}.$$

The following assertion gives some bound on the effective size of $B(\hat{\theta}, M\hat{r}_\kappa)$ in terms of the local radial rate $r(\theta_0)$ from the P_{θ_0} -perspective.

Proposition 3. For a $\theta_0 \in \Theta$, let a DDM $P(\cdot|X)$ and a DD-center $\hat{\theta}$ satisfy (A1) for some radial rate $r(\theta_0)$. Let \hat{r}_κ be defined by (4). Then for any $\kappa \in (0, 1)$, $M > 0$,

$$P_{\theta_0}(\hat{r}_\kappa \geq Mr(\theta_0)) \leq \frac{\phi_1(M)}{\kappa}.$$

Proof. By the conditional Markov inequality, (4) and condition (A1),

$$\begin{aligned} P_{\theta_0}(\hat{r}_\kappa \geq Mr(\theta_0)) &\leq P_{\theta_0}(P(d(\theta, \hat{\theta}) \leq Mr(\theta_0)|X) \leq 1 - \kappa) \\ &= P_{\theta_0}(P(d(\theta, \hat{\theta}) > Mr(\theta_0)|X) > \kappa) \\ &\leq \frac{E_{\theta_0}(P(d(\theta, \hat{\theta}) > Mr(\theta_0)|X))}{\kappa} \leq \frac{\phi_1(M)}{\kappa}. \end{aligned} \quad \square$$

Suppose conditions (A1)–(A3) are fulfilled for some DDM $P(\cdot|X)$ and DD-center $\hat{\theta}$, with some local radial rate $r(\theta_0)$ and functions ϕ_1, ψ, ϕ_2 . Let us elucidate what else is needed in concrete situations to derive the optimality framework (1). Suppose the following uniform bounds hold:

$$\phi_1(M, \varepsilon, \theta_0) \leq \bar{\phi}_1(M) \quad \forall \theta_0 \in \Theta_{size} \subseteq \Theta,$$

$$\phi_2(M, \varepsilon, \theta_0) \leq \bar{\phi}_2(M), \quad \psi(M, \varepsilon, \theta_0) \leq \bar{\psi}(M), \quad \forall \theta_0 \in \Theta_{cov} \subseteq \Theta,$$

for all $\varepsilon \in (0, \varepsilon_0]$, where $\bar{\phi}_1(M) \downarrow 0$, $\bar{\phi}_2(M) \downarrow 0$ as $M \uparrow \infty$ and $\bar{\psi}(\delta) \downarrow 0$ as $\delta \downarrow 0$. Clearly, then Propositions 2 and 3 ensure (1) for the ball $B(\hat{\theta}, M\hat{r}_\kappa)$ and the radial rate $r(\theta_0)$ by

taking sufficiently large M . In fact, we can optimize the choice of M as follows: first determine

$$\min_{\delta > 0} \left\{ \bar{\phi}_2(M\delta) + \frac{\bar{\psi}(\delta)}{1 - \kappa} \right\} = \bar{\phi}(M, \kappa),$$

where $\bar{\phi}(M, \kappa) \downarrow 0$ as $M \uparrow \infty$. Next, take constants M_1 and M_2 sufficiently large so that $\bar{\phi}(M_1, \kappa) \leq \alpha_1$ and $\bar{\phi}_1(M_2)/\kappa \leq \alpha_2$. Then the optimality framework (1) holds with $C = M_1$ and $c = M_2$.

Finally, let us mention some additional material provided in Supplement.

- Two examples, the normal model and the so called *Bernstein-von Mises* case, demonstrating the application of Propositions 2 and 3.
- A corollary from Propositions 1–3 for the default confidence ball $\tilde{B}_{M, \kappa}$ defined by (8), which can also be used for establishing the optimality framework (1).
- A proposition, demonstrating that (A2) is in some sense the minimal condition for providing a sufficient P_{θ_0} -coverage of the $P(\cdot|X)$ -credible ball with the sharpest rate.

3 Inverse signal-in-white-noise model

3.1 The model

Let $\mathbb{N} = \{1, 2, \dots\}$ and $\sigma = (\sigma_i, i \in \mathbb{N})$ be a positive nondecreasing sequence. We observe

$$X = X^{(\varepsilon)} = (X_i, i \in \mathbb{N}) \sim P_\theta = P_\theta^{(\varepsilon)} = \bigotimes_{i \in \mathbb{N}} N(\theta_i, \sigma_i^2), \quad \sigma_i^2 = \varepsilon^2 \kappa_i^2, \quad (9)$$

i.e., $X_i \stackrel{\text{ind}}{\sim} N(\theta_i, \sigma_i^2)$, $i \in \mathbb{N}$. Here $\theta = (\theta_i, i \in \mathbb{N}) \in \Theta = \ell_2$ is an unknown parameter of interest. Without loss of generality, we set

$$\varepsilon^2 = \min_i \sigma_i^2 = \sigma_1^2 \quad \text{and} \quad \kappa_i = \sigma_i / \varepsilon \geq 1, \quad \text{so that} \quad \sigma_i^2 = \varepsilon^2 \kappa_i^2.$$

Thus, the nondecreasing sequence $\{\kappa_i^2, i \in \mathbb{N}\}$ reflects the ill-posedness of the model and ε^2 is the noise intensity describing the information increase in the data $X^{(\varepsilon)}$ as $\varepsilon \rightarrow 0$. The model (9) is known to be the sequence version of the *inverse signal-in-white-noise model*. There is now a vast literature about this model, especially for the direct case: $\kappa_i^2 = 1$, $i \in \mathbb{N}$. This model is of a canonical type and serves, by virtue of the so called *equivalence principle*, as a purified approximation to some other statistical models. The direct case of the model (9) can be related, in exact terms, to the generalized linear Gaussian model as introduced by [4], the continuous white noise model, certain discrete regression model; and as an approximating model, to the density estimation problem, spectral function estimation, various regression models. Examples of inverse problems fitting the framework (9) can be found in [8]; see further references therein. The statistical inference results for the generic model (9) can be conveyed to other models, according to the equivalence principle. However, in general the problem of establishing the equivalence

in a precise sense is a delicate task. We will not go into this, but focus on the model (9). Some more information can be found in Supplement.

By default, all summations and products are over \mathbb{N} , unless otherwise specified, e.g., $\bigotimes_i = \bigotimes_{i \in \mathbb{N}}$. Introduce some notations: $\|\theta\| = (\sum_i \theta_i^2)^{1/2}$ is the ℓ_2 -norm; for $a, b \in \mathbb{R}$, $\lfloor a \rfloor = \max\{z \in \mathbb{Z} : z \leq a\}$, $\Sigma(a) = \sum_{i \leq a} \sigma_i^2$, $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$; $\varphi(x, \mu, \sigma^2)$ is the $N(\mu, \sigma^2)$ -density at x , $N(\mu, 0)$ means a Dirac measure at μ ; the indicator function $1\{E\} = 1$ if the event E occurs and is zero otherwise. Let $\sum_{i=k}^n a_i = 0$ if $n < k$. If random quantities appear in a relation, then this relation should be understood in P_{θ_0} -almost sure sense, for the “true” $\theta_0 \in \Theta$.

We complete this subsection with conditions on σ_i^2 's (or, equivalently, on κ_i^2 's): for any $\rho, \tau_0 \geq 1$, $\gamma > 0$, there exist some positive K_1 , $K_2 = K_2(\rho)$, $K_3 = K_3(\gamma)$, $K_4 \in (0, 1)$, $\tau > 2$ (this can be relaxed to $\tau \geq 1$) and $K_5 = K_5(\tau_0)$ such that the relations

$$\begin{aligned} (i) \quad n\sigma_n^2 &\leq K_1 \Sigma(n), \quad (ii) \quad \Sigma(\rho n) \leq K_2(\rho) \Sigma(n), \\ (iii) \quad \sum_n e^{-\gamma n} \Sigma(n) &\leq K_3(\gamma) \sigma_1^2, \\ (iv) \quad \Sigma(\lfloor m/\tau \rfloor) &\leq (1 - K_4) \Sigma(m), \quad (v) \quad l \sigma_{\lfloor l/\tau_0 \rfloor}^2 \geq K_5(\tau_0) \sum_{i=\lfloor l/\tau_0 \rfloor+1}^l \sigma_i^2, \end{aligned} \tag{10}$$

hold for all $n \in \mathbb{N}$, all $m \geq \tau$ and all $l \geq \tau_0$. Although there is in principle some freedom in choosing sequence κ_i describing the ill-posedness of the problem, to avoid unnecessary technical complications, from now on we assume the so called *mildly ill-posed* case: $\kappa_i^2 = i^{2p}$, $i \in \mathbb{N}$, for some $p \geq 0$.

Remark 2. The mildly ill-posed case $\kappa_i^2 = i^{2p}$ satisfies (10) with $K_1 = 2p + 1$, $K_2 = (\rho + 1)^{2p+1}$, $K_3 = \frac{4(8p+4)^{2p}}{(e\gamma)^{2p+1}(e^{\gamma/2}-1)}$ (a rough bound), $K_4 = \frac{1}{2}$, τ can be any number satisfying $\tau \geq 2^{1+1/(2p+1)}$ and $K_5 = (2\tau_0)^{-2p}$; see Supplement for the calculations.

3.2 Constructing DDM $P(\theta|X)$ as empirical Bayes posterior

Here we construct a DDM $P(\cdot|X)$ on θ which we later use for constructing a confidence set as DDM-credible ball, according to the general approach described in Section 2. The optimality (1) will then be established for appropriate choices of involved quantities.

For some fixed $K, \alpha > 0$, introduce the following (mixture) DDM on θ :

$$P(\cdot|X) = P_{K,\alpha}(\cdot|X) = \sum_I P_I(\cdot|X) P(\mathcal{I} = I|X), \tag{11}$$

where the family of DDMs $\{P_I(\cdot|X), I \in \mathbb{N}\}$ on θ and the DDM $P(\mathcal{I} = I|X)$ on I are

$$P_I(\cdot|X) = \bigotimes_i N(X_i(I), L\sigma_i^2 1\{i \leq I\}), \tag{12}$$

$$P(\mathcal{I} = I|X) = \frac{\lambda_I \bigotimes_i \varphi(X_i, X_i(I), \tau_i^2(I) + \sigma_i^2)}{\sum_J \lambda_J \bigotimes_i \varphi(X_i, X_i(J), \tau_i^2(J) + \sigma_i^2)}, \tag{13}$$

with $L = \frac{K}{K+1}$ (can be any positive value), $C_\alpha = e^\alpha - 1$, and

$$X_i(I) = X_i 1\{i \leq I\}, \tau_i^2(I) = K\varepsilon^2 1\{i \leq I\}, \lambda_I = C_\alpha e^{-\alpha I}, i, I \in \mathbb{N}, \quad (14)$$

so that $\sum_I \lambda_I = 1$. The quantity (13) exists as P_{θ_0} -almost sure limit of

$$P_n(\mathcal{I} = I|X) = \frac{\lambda_I \bigotimes_{i=1}^n \varphi(X_i, X_i(I), \tau_i^2(I) + \sigma_i^2)}{\sum_J \lambda_J \bigotimes_{i=1}^n \varphi(X_i, X_i(J), \tau_i^2(J) + \sigma_i^2)}.$$

The DDM (11) can be associated with the empirical Bayes posterior originating from the following two-level hierarchical prior Π :

$$\theta|\mathcal{I} = I \sim \Pi_{I, \mu(I)} = \bigotimes_i N(\mu_i(I), \tau_i^2(I)), \quad P(\mathcal{I} = I) = \lambda_I, \quad (15)$$

where $\tau_i^2(I)$ and λ_I are defined by (14), $\mu(I) = (\mu_i(I), i \in \mathbb{N})$ with $\mu_i(I) = \mu_{I,i} 1\{i \leq I\}$. Indeed, the model (9) and the prior (15) lead to the corresponding marginal $P_{X, \mu}(X)$ and the posterior $\Pi_\mu(\cdot|X) = \sum_I \Pi_\mu(\cdot|X, \mathcal{I} = I) \Pi_\mu(\mathcal{I} = I|X)$, where $\mu = (\mu(I), I \in \mathbb{N})$ (mind that μ is a sequence of sequences). Then $P(\cdot|X) = \Pi_{\hat{\mu}}(\cdot|X)$, where $L = \frac{K}{K+1}$ and $\hat{\mu} = (\hat{\mu}(I), I \in \mathbb{N})$ (with $\hat{\mu}(I) = (\hat{\mu}_i(I), i \in \mathbb{N})$ and $\hat{\mu}_i(I) = X_i 1\{i \leq I\}$) is the empirical Bayes estimator obtained by maximizing the marginal $P_{X, \mu}(X)$ with respect to μ . Indeed, as is easy to see, $P_I(\cdot|X) = \Pi_{\hat{\mu}}(\cdot|X, \mathcal{I} = I)$ and $P(\mathcal{I} = I|X) = \Pi_{\hat{\mu}}(\mathcal{I} = I|X)$.

Notice that we actually follow the Bayesian tradition since the obtained DDM $P(\cdot|X)$, defined by (11), results from certain empirical Bayes posterior. However, in principle we can manipulate with different ingredient in constructing DDMs. For example, different choices for $P_I(\cdot|X)$ and $P(\mathcal{I} = I|X)$ in (11) are possible, not necessarily coming from the (same) Bayesian approach. For example, some other $P(\mathcal{I} = I|X)$ in (11) will do the job as well, another constant $L > 0$ is possible, etc. More on this is in Supplement.

Remark 3. One more choice for DDM within Bayesian tradition is the empirical Bayes posterior with respect to I :

$$\hat{P}(\cdot|X) = P_{\hat{I}}(\cdot|X), \quad \text{with} \quad \hat{I} = \min \left\{ \arg\max_{I \in \mathbb{N}} P(\mathcal{I} = I|X) \right\}, \quad (16)$$

where $P_I(\cdot|X)$ and $P(\mathcal{I} = I|X)$ are defined by respectively (12) and (13). All the below claims about the DDM $P(\cdot|X)$ defined by (11) hold also for the DDM $\hat{P}(\cdot|X)$ exactly in the same way; see Supplement. A connection of the DDM $\hat{P}(\cdot|X)$ to penalized estimators is also discussed in Supplement.

3.3 Local DDM-contraction rate: upper bound

First we introduce the local contraction rate for the DDM $P(\cdot|X)$. Notice that the DDM $P(\cdot|X)$ is a random mixture over DDMs $P_I(\cdot|X)$, $I \in \mathbb{N}$. From the P_{θ_0} -perspective, each $P_I(\cdot|X)$ contracts to the true θ_0 with the local rate $r(I, \theta_0)$:

$$r^2(I, \theta_0) = r_\sigma(I, \theta_0) = \sum_{i \leq I} \sigma_i^2 + \sum_{i > I} \theta_{0,i}^2, \quad I \in \mathbb{N}. \quad (17)$$

Indeed, denoting $X(I) = (X_i 1\{i \leq I\}, i \in \mathbb{N})$, we evaluate

$$\begin{aligned} \mathbb{E}_{\theta_0} P_I(\|\theta - \theta_0\| \geq Mr(I, \theta_0) | X) &\leq \frac{\mathbb{E}_{\theta_0} [\|X(I) - \theta_0\|^2 + L \sum_{i \leq I} \sigma_i^2]}{M^2 r^2(I, \theta_0)} \\ &= \frac{2 \sum_{i \leq I} \sigma_i^2 + \sum_{i > I} \theta_{0,i}^2}{M^2 r^2(I, \theta_0)} \leq \frac{2}{M^2}. \end{aligned} \quad (18)$$

Thus, we have the family of local rates $\mathcal{P} = \mathcal{P}(\mathbb{N}) = \{r(I, \theta), I \in \mathbb{N}\}$. For each $\theta \in \ell_2$, there is the best choice $I_o = I_o(\theta) = I_o(\theta, \sigma)$ of parameter I , called *oracle*, corresponding to the smallest possible rate $r(I_o, \theta)$ called the *oracle rate* (over the family \mathcal{P}) given by

$$r^2(\theta) = r^2(I_o, \theta) = \min_{I \in \mathbb{N}} r^2(I, \theta) = \sum_{i \leq I_o} \sigma_i^2 + \sum_{i > I_o} \theta_i^2. \quad (19)$$

Notice $r^2(\theta) \geq \sigma_1^2 = \varepsilon^2$ and $I_o(\theta) \geq 1$ for any $\theta \in \ell_2$, because we minimize over \mathbb{N} . This is not restrictive since if the minimum is taken over $I \in \mathbb{N} \cup \{0\}$, all the results will hold only for the oracle rate with an additive penalty term, a multiple of ε^2 . This will boil down to the same resulting local rate.

The following theorem establishes the local upper bound (19) for the contraction rate of the DDM $P(\cdot | X)$ defined by (11).

Theorem 1 (Upper bound). *Let the DDM $P(\cdot | X)$ and the local rate $r(\theta)$ be defined by (11) and (19) respectively, with $K \geq 1.87$, $\alpha > 0$. Then there exists a constant $C_{or} = C_{or}(K, \alpha)$ such that, for any $\theta_0 \in \ell_2$ and $M > 0$,*

$$\mathbb{E}_{\theta_0} P(\|\theta - \theta_0\| \geq Mr(\theta_0) | X) \leq \frac{C_{or}}{M^2}.$$

We provide the proof of this theorem in Section 4. Although the condition $K \geq 1.87$ emerges as an artifact of the proof technique, in a way it has the same meaning as bounds for the penalty constants for the penalized estimators. More on this is in Supplement.

Theorem 1 establishes a non-asymptotic local upper bound for the contraction rate of the DDM (11) for the model (9), uniformly over ℓ_2 -space. This ensures the size property in (1) for the default confidence ball (8) by using the DDM (11), with the radial rate $r(\theta_0)$ defined by (19) and $\Theta_{size} = \ell_2$. We will come back to this when proving the main result, Theorem 4.

Besides being an ingredient for establishing the confidence optimality (1), the above theorem is of its own interest. The results with local contraction rates are intrinsically adaptive in the sense that the contraction rate $r(I_o, \theta_0)$ is fast for “smooth” θ_0 ’s and slow for “rough” ones. This is a stronger and more refined property than being globally adaptive. Let us elucidate the potential strength of local results.

To characterize the quality of Bayesian procedures, the notion of posterior contraction rate was first introduced and studied in [12]. Clearly, it extends directly to DDMS and a non-asymptotic version of this notion for DDMS is in fact given by condition ($\tilde{A}2$). Typically in the literature, contraction rate is related to the (global) minimax rate $R(\Theta_\beta)$ over a certain set $\Theta_\beta \ni \theta_0$. The optimality of Bayesian procedures is then understood in

the sense of adaptive minimax posterior convergence rate: given a prior (knowledge of β is not used in the prior), the resulting posterior contracts, from the P_{θ_0} -perspective, to the “true” $\theta_0 \in \Theta_\beta$ with the minimax rate $R(\Theta_\beta)$.

For a scale $\Theta(\mathcal{B}) = \{\Theta_\beta, \beta \in \mathcal{B}\}$, let $\{R(\Theta_\beta), \beta \in \mathcal{B}\}$ be the family of the pertaining minimax rates. Suppose (3) is fulfilled for the local rate $r(\theta_0)$ defined by (19) and $\{R(\Theta_\beta), \beta \in \mathcal{B}\}$. Then, in view of (3), Theorem 1 entails that the DDM $P(\cdot|X)$ (11) must also contract to θ_0 with (at least) the minimax rate $R_\varepsilon(\Theta_\beta)$ uniformly in $\theta_0 \in \Theta_\beta$ for each $\beta \in \mathcal{B}$. Thus, the adaptive (over the scale $\Theta(\mathcal{B})$) minimax contraction rate result for $P(\cdot|X)$ follows immediately. Foremost, Theorem 1 implies adaptive minimax results *simultaneously for all scales* for which (3) is fulfilled. In particular, (3) is satisfied for the following scales: Sobolev and analytic ellipsoids, Sobolev hyperrectangles (in fact, rather general ℓ_2 -ellipsoids and hyperrectangles, considered below), certain scales of Besov classes and ℓ_p -bodies, tail classes. See Supplement for details where we also consider the situation when the local oracle results over one family of rates imply the local oracle results over another family of rates.

For example, consider general ellipsoids and hyperrectangles

$$\mathcal{E}(a) = \{\theta \in \ell_2 : \sum_i (\frac{\theta_i}{a_i})^2 \leq 1\}, \quad \mathcal{H}(a) = \{\theta \in \ell_2 : |\theta_i| \leq a_i, i \in \mathbb{N}\}, \quad (20)$$

where $a = (a_i, i \in \mathbb{N})$ is nonincreasing sequence of numbers in $[0, +\infty]$ which converge to 0 as $i \rightarrow \infty$, $a_1 \geq c_1 \varepsilon$ for some $c_1 > 0$. Here we adopt the conventions $0/0 = 0$ and $x/(+\infty) = 0$ for $x \in \mathbb{R}$. Let $R^2(\Theta) = \inf_{\hat{\theta}} \sup_{\theta \in \Theta} E_\theta \|\hat{\theta} - \theta\|^2$ denote the (quadratic) minimax risk over a set Θ , where the infimum is taken over all possible estimators $\hat{\theta} = \hat{\theta}(X)$, measurable functions of the data X . One can show (see Supplement) that

$$\sup_{\theta_0 \in \mathcal{E}(a)} r^2(\theta_0) \leq (2\pi)^2 R^2(\mathcal{E}(a)), \quad \sup_{\theta_0 \in \mathcal{H}(a)} r^2(\theta_0) \leq \frac{5}{2} R^2(\mathcal{H}(a)). \quad (21)$$

Instead of $(2\pi)^2$, one can put a tighter constant 4.44 in the direct case, which possibly holds for the ill-posed case as well; see Supplement. Then Theorem 1 implies that

$$\sup_{\theta_0 \in \Theta(a)} E_{\theta_0} P(\|\theta - \theta_0\| \geq MR(\Theta(a)) | X) \leq \frac{C}{M^2}, \quad \text{for some } C = C(K, \alpha),$$

where $\Theta(a)$ is either $\mathcal{E}(a)$ or $\mathcal{H}(a)$ for some *unknown* a . In particular, we obtain the minimax contraction rates for the four scales considered in [21]: two families of ellipsoids (Sobolev and analytic) and two families of hyperrectangles (Sobolev and parametric); see Supplement.

It turns out that the DDM $P(\cdot|X)$ defined by (11) can also be used for estimating the parameter θ_0 . Namely, define the estimator

$$\tilde{\theta} = E(\theta|X) = \sum_I X(I) P(\mathcal{I} = I|X), \quad X(I) = (X_i 1\{i \leq I\}, i \in \mathbb{N}), \quad (22)$$

which is just the DDM $P(\cdot|X)$ -expectation. This estimator satisfies the following oracle estimation inequality.

Theorem 2 (Oracle inequality). *Let the conditions of Theorem 1 be fulfilled, $\theta_0 \in \ell_2$, and $\tilde{\theta}$ be defined by (22). Then there exist a constant $C_{est} = C_{est}(K, \alpha) \geq 1$ such that $E_{\theta_0} \|\tilde{\theta} - \theta_0\|^2 \leq C_{est} r^2(\theta_0)$, where the oracle rate $r(\theta)$ is defined by (19).*

This theorem yields the whole panorama of the minimax adaptive estimation results in the mildly ill-posed inverse setting, simultaneously over all scales for which (3) is fulfilled. The proof of this theorem is essentially contained in the proof of Theorem 1, but it is still provided in Supplement. Similar result has been obtained in [9] for the estimator based on the risk hull minimization method. In that paper, the oracle rate has an extra penalty term but the multiplicative constant is very tight.

3.4 Local DDM-contraction rate: lower bound under EBR

As we mentioned in the introduction, in general it is impossible to construct optimal (fully) adaptive confidence set in the minimax sense with a prescribed high coverage probability. Actually, this is a genuine problem, not connected with an optimality framework used: global minimax (2), or local (1). Clearly, the same problem should occur for the local approach (1), because otherwise we would have solved the minimax version of the problem as well. The intuition is that there are so called “deceptive” parameters θ_0 that “trick” the DDM $P(\cdot|X)$ in the sense that the random radius \hat{r} defined by (4) is overoptimistic, i.e., of a smaller order than the actual radial rate $r(\theta_0)$. The coverage probability is then too small.

A way to fix this problem is to remove a set (preferably, minimal) of deceptive parameters from the set Θ (in our case ℓ_2) and derive the coverage relation in (1) for the remaining set of non-deceptive parameters. In a different framework, [18] introduced such a set, the so called *self-similar* (SS) parameters, studied later by many authors in various settings and models. A somewhat restrictive feature of the self-similarity property is that it is linked to the Sobolev (Besov) smoothness scale. In [21] a more general condition is introduced that is not linked to a particular smoothness scale, the *polished tail* (PT) condition: for some $L_0 > 0$ ($L_0 \geq 1$ for Θ_{pt} to be not empty), $N_0 \in \mathbb{N}$ and $\rho_0 \geq 2$,

$$\Theta_{pt} = \Theta_{pt}(L_0, N_0, \rho_0) = \left\{ \theta \in \ell_2 : \sum_{i=N}^{\infty} \theta_i^2 \leq L_0 \sum_{i=N}^{\rho_0 N} \theta_i^2, \forall N \geq N_0 \right\}.$$

In [21] it is shown that $\Theta_{ss} \subseteq \Theta_{pt}$, i.e., PT is more general than SS.

Introduce the *surrogate oracle rate* $r(\bar{I}_o, \theta_0)$, with the *surrogate oracle* \bar{I}_o defined as follows:

$$\bar{I}_o = \operatorname{argmin}_I R^2(I, \theta_0), \quad R^2(I, \theta_0) = R_{\sigma}^2(I, \theta_0) = I\varepsilon^2 + \sum_{i>I} \frac{\theta_{0,i}^2}{\kappa_i^2}. \quad (23)$$

The quantity $R(I, \theta_0)$ is nothing else but the oracle rate for the parameter $\bar{\theta} = (\theta_i/\kappa_i, i \in \mathbb{N})$ in the “direct” model $\tilde{X} = (X_i/\kappa_i, i \in \mathbb{N}) \sim \bigotimes_i N(\bar{\theta}_i, \varepsilon^2)$. Note that $\bar{I}_o = I_o$ in the direct case $\kappa_i^2 = 1$.

Introduce the *excessive bias restriction* (EBR): $\theta_0 \in \Theta_{eb}(\tau)$ for $\tau > 0$,

$$\Theta_{eb} = \Theta_{eb}(\tau) = \Theta_{eb}(\tau, \varepsilon) = \left\{ \theta \in \ell_2 : \sum_{i > \bar{I}_o} \theta_i^2 \leq \tau \sum_{i \leq \bar{I}_o} \sigma_i^2 \right\},$$

where $\bar{I}_o = \bar{I}_o(\theta)$ is defined by (23). Note that in principle Θ_{eb} also depends on ε as we consider the non-asymptotic setting. For asymptotic considerations (as $\varepsilon \rightarrow 0$), we can introduce a uniform (in ε) version of EBR:

$$\bar{\Theta}_{eb}(\tau, \varepsilon_0) = \left\{ \theta \in \Theta_{eb}(\tau, \varepsilon) \text{ for all } \varepsilon \in (0, \varepsilon_0] \right\} = \cap_{\varepsilon \in (0, \varepsilon_0]} \Theta_{eb}(\tau, \varepsilon).$$

We will not consider $\bar{\Theta}_{eb}(\tau, \varepsilon_0)$ and $\Theta_{eb}(\tau, \varepsilon)$ separately and will always use the latter notation $\Theta_{eb}(\tau)$ for both in what follows, with the understanding that whenever one needs the uniform version, one can think of $\Theta_{eb}(\tau)$ as $\bar{\Theta}_{eb}(\tau, \varepsilon_0)$, as all assertions below hold also for the uniform version of EBR.

Let us show that EBR is less restrictive than PT, i.e., for any $L_0 \geq 1$, $N_0 \in \mathbb{N}$ and $\rho_0 \geq 2$, there exists a $\tau > 0$ such that $\Theta_{pt}(L_0, N_0, \rho_0) \subseteq \Theta_{eb}(\tau)$. From (23), it follows that for any $I > \bar{I}_o$, $\sum_{i=\bar{I}_o+1}^I \frac{\theta_i^2}{\sigma_i^2} \leq I - \bar{I}_o$. Besides, by condition (i) in (10), $(n-l)\sigma_n^2 \leq K_1 \sum_{i=l+1}^n \sigma_i^2$ (indeed, σ is non-decreasing and $K_1 \leq 1$) for all $n, l \in \mathbb{N}$ such that $n > l$. Using the last two relations and the property (ii) from (10), we obtain for any $\theta \in \Theta_{pt}(L_0, N_0, \rho_0)$ that

$$\begin{aligned} \sum_{i=\bar{I}_o+1}^{\infty} \theta_i^2 &= \sum_{i=\bar{I}_o+1}^{N_0 \bar{I}_o - 1} \theta_i^2 + \sum_{i=N_0 \bar{I}_o}^{\infty} \theta_i^2 \leq \sum_{i=\bar{I}_o+1}^{N_0 \bar{I}_o - 1} \theta_i^2 + L_0 \sum_{i=N_0 \bar{I}_o}^{\rho_0 N_0 \bar{I}_o} \theta_i^2 \\ &\leq L_0 \sigma_{\rho_0 N_0 \bar{I}_o}^2 \sum_{i=\bar{I}_o+1}^{\rho_0 N_0 \bar{I}_o} \frac{\theta_i^2}{\sigma_i^2} \leq L_0 \sigma_{\rho_0 N_0 \bar{I}_o}^2 (\rho_0 N_0 \bar{I}_o - \bar{I}_o) \\ &\leq L_0 K_1 \sum_{i=\bar{I}_o+1}^{\rho_0 N_0 \bar{I}_o} \sigma_i^2 \leq L_0 K_1 \sum_{i=1}^{\rho_0 N_0 \bar{I}_o} \sigma_i^2 \leq L_0 K_1 K_2 (\rho_0 N_0) \sum_{i=1}^{\bar{I}_o} \sigma_i^2, \end{aligned}$$

so that $\Theta_{pt}(L_0, N_0, \rho_0) \subseteq \Theta_{eb}(L_0 K_1 K_2 (\rho_0 N_0))$ for any $N_0 \geq 1$.

Summarizing the relations between three types of conditions describing non-deceptive parameters introduced above, $\Theta_{ss} \subseteq \Theta_{pt} \subseteq \Theta_{eb}$. Thus EBR is the most general condition among these three. As to the question how big (or “typical”) that set Θ_{eb} is, [21] gives three types of arguments for the PT-parameters: topological, minimax and Bayesian. Since $\Theta_{eb} \supseteq \Theta_{pt}$, the same arguments certainly apply to Θ_{eb} ; see [21] for more details on this.

Now we are ready to formulate the lower bound result for the DDM-contraction rate.

Theorem 3 (Small ball DDM-probability). *Let the DDM $P(\cdot|X) = P_{K,\alpha}(\cdot|X)$ be given by (11), with parameters $K, \alpha > 0$ such that*

$$\alpha < a(K) \triangleq \frac{1}{4} - \frac{1}{2} \log \left(\frac{K+1}{2} \right). \quad (24)$$

Then there exists $C_{sb} = C_{sb}(K, \alpha) > 0$ such that, for any $\theta_0 \in \ell_2$, any DD-center $\hat{\theta} = \hat{\theta}(X)$ and any $\delta \in (0, \delta_{sb}]$ with $\delta_{sb} = 1 \wedge \left(\sqrt{\frac{K(2p+1)}{K+1}} \left(\frac{a(K)-\alpha}{4ea(K)} \right)^{p+\frac{1}{2}} \right)$,

$$\mathbb{E}_{\theta_0} \mathbb{P}(\|\theta - \hat{\theta}\| \leq \delta \Sigma^{1/2}(\bar{I}_o) | X) \leq C_{sb} \delta [\log(\delta^{-1})]^{p+1/2},$$

where $\Sigma(\bar{I}_o) = \sum_{i \leq \bar{I}_o} \sigma_i^2$ and $\bar{I}_o = \bar{I}_o(\theta_0)$ is defined by (23).

Notice that the effective rate in the above lower bound is determined by the variance term of the oracle surrogate rate $r^2(\bar{I}_o, \theta_0)$. Recall that EBR says basically that the variance term is the main term in the surrogate oracle rate $r^2(\bar{I}_o, \theta_0)$. Under $\theta_0 \in \Theta_{eb}(\tau)$, we thus have $r^2(\theta_0) = r^2(I_o, \theta_0) \leq r^2(\bar{I}_o, \theta_0) \leq (1 + \tau) \Sigma(\bar{I}_o)$. This yields the following corollary.

Corollary 1 (Lower bound under EBR). *Let the conditions of Theorem 3 be satisfied. Then, for any $\theta_0 \in \ell_2$, any DD-center $\hat{\theta} = \hat{\theta}(X)$, any $\tau > 0$ and any $\delta \in (0, \delta_{eb}]$ with $\delta_{eb} = (1 + \tau)^{-1/2} \delta_{sb}$,*

$$\sup_{\theta_0 \in \Theta_{eb}(\tau)} \mathbb{E}_{\theta_0} \mathbb{P}(\|\theta - \hat{\theta}\| \leq \delta r(\theta_0) | X) \leq C_{eb} \delta [\log(\delta^{-1})]^{p+1/2},$$

where $C_{eb} = C_{sb} \sqrt{1 + \tau}$, δ_{sb} and C_{sb} are from Theorem 3.

A bound (not the sharpest) for the constant C_{sb} can be found in the proof of Theorem 3. The above assertion implies condition (A2) for the DDM $\mathbb{P}(\cdot | X)$ satisfying the conditions of Theorem 3, with $\psi(\delta) = C_{eb} \delta [\log(\delta^{-1})]^{p+\frac{1}{2}}$, uniformly in $\theta_0 \in \Theta_{eb}(\tau)$. This ensures the coverage relation in (1), see the next subsection.

3.5 The main result: confidence ball under EBR

In this subsection we establish the main result of the paper. Let the DDM $\mathbb{P}(\cdot | X) = \mathbb{P}_{K,\alpha}(\cdot | X)$ be given by (11), with constants $K, \alpha > 0$ such that $K > 1.87$ and condition (24) is fulfilled. By using the DDM $\mathbb{P}(\cdot | X)$, we construct the default DD-center $\check{\theta}$ (7) and the default confidence ball $\tilde{B} = B(\check{\theta}, M\hat{r}_\kappa)$ given by (8), with some fixed $\kappa \in (0, 1)$, say $\kappa = \frac{1}{2}$. Theorem 1 implies (A1) with $\varphi(M) = \frac{C_{or}}{M^2}$. Then by Proposition 1, (A1) and (A3) are also fulfilled for the DDM $\mathbb{P}(\cdot | X)$ and the default DD-center $\check{\theta}$, with (see Remark 1) $\phi_1(M) = \frac{42C_{or}}{M^2}$, $\phi_2(M) = \frac{10C_{or}}{M^2}$, uniformly in $\theta_0 \in \ell_2$.

Let us bound the coverage probability of the default confidence ball $B(\check{\theta}, M\hat{r}_\kappa)$. In view of Corollary 1, we conclude that condition (A2) is met with $\psi(\delta) = C_{eb} \delta [\log(\delta^{-1})]^{p+1/2}$, uniformly in $\theta_0 \in \Theta_{eb}(\tau)$. As also (A3) is fulfilled with $\phi_2(M) = \frac{10C_{or}}{M^2}$ uniformly in $\theta_0 \in \ell_2 \supseteq \Theta_{eb}$, by applying Proposition 2 we derive that, for each $\theta_0 \in \Theta_{eb}(\tau)$,

$$\mathbb{P}_{\theta_0}(\theta_0 \notin B(\check{\theta}, M\hat{r}_\kappa)) \leq \phi_2(M\delta) + \frac{\psi(\delta)}{1 - \kappa} = \frac{10C_{or}}{M^2\delta^2} + \frac{C_{eb}\delta [\log(\delta^{-1})]^{p+\frac{1}{2}}}{1 - \kappa}$$

for any $M, \delta > 0$. For $\alpha_1 \in (0, 1)$ and δ_{eb} defined in Corollary 1, we take

$$\delta_1 = \max \{ \delta \in (0, \delta_{eb}] : C_{eb}(1 - \kappa)^{-1} \delta [\log(\delta^{-1})]^{p+1/2} \leq \alpha_1/2 \}$$

and $M_1 = \min\{M \in \mathbb{N} : 10C_{or}/(M\delta_1)^2 \leq \alpha_1/2\}$. Then, for all $M \geq M_1$,

$$\sup_{\theta_0 \in \Theta_{eb}(\tau)} P_{\theta_0}(\theta_0 \notin B(\check{\theta}, M\hat{r}_\kappa)) \leq \alpha_1. \quad (25)$$

Now, since condition (A1) is satisfied with $\phi_1(M) = \frac{42C_{or}}{M^2}$, applying Proposition 3 yields that the size \hat{r}_κ of the confidence ball $B(\check{\theta}, M\hat{r}_\kappa)$ is of the local radial rate order:

$$P_{\theta_0}(\hat{r}_\kappa \geq Mr(\theta_0)) \leq \frac{\phi_1(M)}{\kappa} = \frac{42C_{or}}{\kappa M^2},$$

for any $M > 0$ and all $\theta_0 \in \ell_2$. For $\alpha_1 \in (0, 1)$, take $M_2 = \min\{M \in \mathbb{N} : 42C_{or}/(\kappa M^2) \leq \alpha_2\}$. Then for any $M \geq M_2$

$$\sup_{\theta_0 \in \ell_2} P_{\theta_0}(\hat{r}_\kappa \geq Mr(\theta_0)) \leq \alpha_2. \quad (26)$$

By combining (25) and (26), we obtain the main result of the paper.

Theorem 4 (Confidence optimality under EBR). *Let the DDM $P(\cdot|X) = P_{K,\alpha}(\cdot|X)$ be given by (11), with constants $K, \alpha > 0$ such that $K \geq 1.87$ and (24) is fulfilled. Further, let $B(\check{\theta}, M\hat{r}_\kappa)$ be the default confidence ball defined by (8). Then for any $\tau > 0$ and any $\alpha_1, \alpha_2 \in (0, 1)$ there exist $C_0 = C_0(\alpha_1, \tau)$ and $c_0 = c_0(\alpha_2)$ such that, for any $C \geq C_0$ and $c \geq c_0$, the following relations hold*

$$\sup_{\theta_0 \in \Theta_{eb}(\tau)} P_{\theta_0}(\theta_0 \notin B(\check{\theta}, C\hat{r}_\kappa)) \leq \alpha_1, \quad \sup_{\theta_0 \in \ell_2} P_{\theta_0}(\hat{r}_\kappa \geq cr(\theta_0)) \leq \alpha_2,$$

where the local radial rate $r(\theta_0)$ is defined by (19).

In the proofs of Theorems 1 and 3, tighter exponential bounds are possible (based on the exponential bounds for the χ^2 -distribution), which would presumably lead to exponential functions φ and ψ in conditions (A1) and (A2). We however use simpler bounds obtained by the Markov inequality for the sake of a succinct presentation. Another useful feature of this approach is that it can be extended to non-normal DDMs (12); see Supplement.

3.6 Concluding remarks

Range for constant K The condition $\alpha < a(K)$ ((24) in Theorem 3) limits room for choosing constants $K, \alpha > 0$, because $a(K) > 0$ only for $K \in (0, 2e^{1/2} - 1)$. One can choose, for example, $K = 2$ and $\alpha = 0.04$. Even less room for K remains if we also want Theorem 1 to hold (and this is needed for the main result, Theorem 4). Indeed, then K has also to satisfy $K \geq 1.87$, so that the final range of allowable K 's becomes $K \in [1.87, 2.29] \subset [1.87, 2e^{1/2} - 1)$. The conditions $K \geq 1.87$ and $\alpha < a(K)$ are apparently more strict than needed for the corresponding theorems to hold, since of course not the most accurate bounds are used in the proof.

Alternative DD-center and confidence ball In Theorem 4, instead of the default DD-center $\check{\theta}$ we can use the estimator $\tilde{\theta}$ defined by (22). Indeed, by Theorems 1 and 2, we have that, uniformly in $\theta_0 \in \ell_2$,

$$\begin{aligned} \mathbb{E}_{\theta_0}[\mathbb{P}(\|\theta - \tilde{\theta}\| \geq Mr(\theta_0)|X)] &\leq \mathbb{E}_{\theta_0}[\mathbb{P}(\|\theta - \theta_0\| \geq \tfrac{1}{2}Mr(\theta_0)|X)] \\ &\quad + \mathbb{E}_{\theta_0}[\mathbb{P}(\|\theta_0 - \tilde{\theta}\| \geq \tfrac{1}{2}Mr(\theta_0)|X)] \leq \frac{4(C_{or} + C_{est})}{M^2} = \phi_1(M), \\ \mathbb{P}_{\theta_0}(\|\theta_0 - \tilde{\theta}\| \geq Mr(\theta_0)) &\leq \frac{\mathbb{E}_{\theta_0}\|\theta_0 - \tilde{\theta}\|^2}{M^2 r^2(\theta_0)} \leq \frac{C_{est}}{M^2} = \phi_2(M). \end{aligned}$$

This means that conditions (A1) and (A3) are also fulfilled for the estimator $\tilde{\theta}$ defined by (22), with $\phi_1(M) = \frac{4(C_{or}+C_{est})}{M^2}$ and $\phi_2(M) = \frac{C_{est}}{M^2}$, uniformly in $\theta_0 \in \ell_2$. Arguing as above, we obtain that Theorem 4 also holds for the DD-center $\tilde{\theta}$ and the confidence ball $B(\tilde{\theta}, M\hat{r}_\kappa)$.

Connection with the minimax results of [21] For the mildly ill-posed inverse signal-in-white-noise model (9), an intriguing paper [21] deals with a certain Sobolev type family of priors, indexed by a smoothness parameter. The proposed DDM is the empirical Bayes posterior with respect to the smoothness parameter. This DDM is then used to construct a DDM-credible ball whose coverage and size properties are studied. The main results of the paper are the asymptotic (in our notation: as $\varepsilon \rightarrow 0$) versions of the minimax framework (2) with $\Theta'_{cov} = \Theta_{pt}$ (the *polished tail* class Θ_{pt} defined in Subsection 3.4), and four choices of scales: Sobolev type scales of hyperrectangles and ellipsoids and the two so called *supersmooth* scales (analytic ellipsoid and parametric hyperrectangle). The proposed DDM is well suited to model Sobolev-type scales: the optimal (minimax) radial rates are obtained in the size relation of (2) for Sobolev hyperrectangles and ellipsoids; but only suboptimal rates are obtained for the two supersmooth scales.

Since $\Theta_{pt} \subseteq \Theta_{eb}$ and the considered four scales are particular examples of scales for which (3) holds, the non-asymptotic versions of the minimax results for all these four scales (including the two supersmooth scales) immediately follow from Theorem 4 for the DDM $\mathbb{P}(\cdot|X)$ (11). Asymptotic versions can readily be derived from the non-asymptotic ones. We emphasize that the scope of the DDM $\mathbb{P}(\cdot|X)$ in delivering the minimax rates extends further than just these four scales. Theorem 4 implies the minimax results of type (2) for all scales for which (3) holds; for example, in view of (21), for all ellipsoids $\mathcal{E}(a)$ and hyperrectangles $\mathcal{H}(a)$ defined by (20). Other smoothness scales can also be treated. Some details are provided in Supplement.

4 Proofs of Theorems 1 and 3

4.1 Proof of Theorem 1

Step 1: bounds for $\mathbb{E}_{\theta_0}\mathbb{P}(\mathcal{I} = I|X)$ For any $I, I_0 \in \mathbb{N}$ and any $h \in [0, 1]$, we have

$$\mathbb{E}_{\theta_0}\mathbb{P}(\mathcal{I} = I|X) \leq \mathbb{E}_{\theta_0} \left[\frac{\lambda_I \otimes_i \varphi(X_i, X_i(I), \tau_i^2(I) + \sigma_i^2)}{\lambda_{I_0} \otimes_i \varphi(X_i, X_i(I_0), \tau_i^2(I_0) + \sigma_i^2)} \right]^h. \quad (27)$$

Recall the elementary identity: for $Y \sim N(\mu, \sigma^2)$ and $b > -\sigma^{-2}$,

$$\mathbb{E}(\exp\{-bY^2/2\}) = \exp\left\{-\frac{\mu^2 b}{2(1+b\sigma^2)} - \frac{1}{2}\log(1+b\sigma^2)\right\}. \quad (28)$$

Using (27) and (28) with $h = 1$, we derive that, for any $I, I_0 \in \mathbb{N}$ such that $I < I_0$,

$$\begin{aligned} \mathbb{E}_{\theta_0} \mathbb{P}(\mathcal{I} = I|X) &\leq \mathbb{E}_{\theta_0} \frac{\lambda_I (K+1)^{-I/2} \exp\left\{-\sum_{i=I+1}^{\infty} \frac{X_i^2}{2\sigma_i^2}\right\}}{\lambda_{I_0} (K+1)^{-I_0/2} \exp\left\{-\sum_{i=I_0+1}^{\infty} \frac{X_i^2}{2\sigma_i^2}\right\}} \\ &= e^{\alpha(I_0-I)} (K+1)^{(I_0-I)/2} \mathbb{E}_{\theta_0} \exp\left\{-\frac{1}{2} \sum_{i=I+1}^{I_0} \frac{X_i^2}{\sigma_i^2}\right\} \\ &= e^{-(\alpha+a_K)I} \exp\left\{(\alpha+a_K)I_0 - \frac{1}{4} \sum_{i=I+1}^{I_0} \frac{\theta_{0,i}^2}{\sigma_i^2}\right\}, \end{aligned} \quad (29)$$

where $a_K = \frac{1}{2} \log(\frac{K+1}{2})$. Now apply (27) and (28) to the case $I > I_0$: for any $h \in [0, 1)$,

$$\begin{aligned} \mathbb{E}_{\theta_0} \mathbb{P}(\mathcal{I} = I|X) &\leq e^{\alpha h(I_0-I)} (K+1)^{(I_0-I)h/2} \mathbb{E}_{\theta_0} \exp\left\{\frac{h}{2} \sum_{i=I_0+1}^I \frac{X_i^2}{\sigma_i^2}\right\} = \\ &= e^{-\alpha h I/2} \exp\left\{-\frac{\alpha h I}{2} + \alpha h I_0 - b_{K,h}(I-I_0) + \frac{h}{2(1-h)} \sum_{i=I_0+1}^I \frac{\theta_{0,i}^2}{\sigma_i^2}\right\}, \end{aligned} \quad (30)$$

where $b_{K,h} = \frac{h}{2} \log(K+1) + \frac{1}{2} \log(1-h)$. Clearly, $b_{K,h} > 0$ if $K > (1-h)^{-1/h} - 1$. Now take $h = 0.1$ in (30), then $b_{K,0.1} = \frac{1}{20} \log(K+1) + \frac{1}{2} \log(0.9) > 0$ since $K \geq 1.87 > (10/9)^{10} - 1$ by the condition of the theorem. Thus, for any $I, I_0 \in \mathbb{N}$ such that $I > I_0$, we derive

$$\mathbb{E}_{\theta_0} \mathbb{P}(\mathcal{I} = I|X) \leq e^{-\alpha I/20} \exp\left\{-\frac{\alpha}{20} \left(I - 2I_0 - \frac{10}{9\alpha} \sum_{i=I_0+1}^I \frac{\theta_{0,i}^2}{\sigma_i^2}\right)\right\}. \quad (31)$$

Step 2: a bound by the sum of three terms Recall $r^2(I, \theta_0) = \sum_{i \leq I} \sigma_i^2 + \sum_{i > I} \theta_{0,i}^2$ and $r^2(\theta_0) = r^2(I_0, \theta_0) = \min_I r^2(I, \theta_0)$. Notice that

$$r^2(I, \theta_0) \leq r^2(\theta_0) + 1\{I \leq I_0\} \sum_{i=I+1}^{I_0} \theta_{0,i}^2 + 1\{I > I_0\} \sum_{i=I_0+1}^I \sigma_i^2. \quad (32)$$

Next, as $\mathbb{P}_I(\cdot|X) = \bigotimes_i N(X_i 1\{i \in \mathbb{N}_I\}, L\sigma_i^2 1\{i \leq I\})$ with $L = \frac{K}{K+1} \leq 1$, we obtain by applying the Markov inequality that

$$\begin{aligned} \mathbb{P}_I(\|\theta - \theta_0\| \geq Mr(\theta_0)|X) &\leq \frac{\mathbb{E}_I(\|\theta - \theta_0\|^2|X)}{M^2 r^2(\theta_0)} \\ &= \frac{L \sum_{i \leq I} \sigma_i^2 + \sum_{i > I} \theta_{0,i}^2 + \sum_{i \leq I} (X_i - \theta_{0,i})^2}{M^2 r^2(\theta_0)} \end{aligned}$$

$$\leq \frac{r^2(I, \theta_0) + \sum_{i \leq I} \sigma_i^2 \xi_i^2}{M^2 r^2(\theta_0)} \triangleq v_I, \quad (33)$$

where $\xi_i = \sigma_i^{-1}(X_i - \theta_{0,i}) \stackrel{\text{ind}}{\sim} N(0, 1)$ from the P_{θ_0} -perspective. Denote for brevity $p_I = P(\mathcal{I} = I|X)$, so that $p_I \in [0, 1]$ and $\sum_I p_I = 1$. In view of (11) and (33),

$$P(\|\theta - \theta_0\| \geq Mr(\theta_0)|X) \leq \sum_I v_I p_I = T_1 + T_2 + T_3, \quad (34)$$

where $T_1 = \sum_{I \leq I_o} v_I p_I$, $T_2 = \sum_{I_o < I \leq \tau I_o} v_I p_I$, $T_3 = \sum_{I > \tau I_o} v_I p_I$, and $\tau > 2$ to be chosen later.

Step 3: handling the term T_1 For $\tau_1 > 0$ to be chosen later, introduce the sets

$$\begin{aligned} \mathcal{O}^- &= \mathcal{O}^-(\tau_1, \theta_0) = \left\{ I : I \leq I_o, \sum_{i=I+1}^{I_o} \theta_{0,i}^2 \leq \tau_1 \sum_{i=1}^{I_o} \sigma_i^2 \right\}, \\ \mathcal{N}^- &= \mathcal{N}^-(\tau_1, \theta_0) = \left\{ I : I \leq I_o, \sum_{i=I+1}^{I_o} \theta_{0,i}^2 > \tau_1 \sum_{i=1}^{I_o} \sigma_i^2 \right\}. \end{aligned}$$

By (32), $\max_{I \in \mathcal{O}^-} r^2(I, \theta_0) \leq (1 + \tau_1)r^2(\theta_0)$. This and (33) imply

$$E_{\theta_0} \sum_{I \in \mathcal{O}^-} v_I p_I \leq E_{\theta_0} \max_{I \in \mathcal{O}^-} v_I \leq \frac{1 + \tau_1}{M^2} + \frac{E_{\theta_0} \sum_{i \leq I_o} \sigma_i^2 \xi_i^2}{M^2 r^2(\theta_0)} \leq \frac{2 + \tau_1}{M^2}. \quad (35)$$

The property (i) from (10) yields that $I_o \leq \frac{K_1}{\sigma_{I_o}^2} \sum_{i=1}^{I_o} \sigma_i^2$. Besides, for each $I \in \mathcal{N}^-$, $\sum_{i=1}^{I_o} \sigma_i^2 < \tau_1^{-1} \sum_{i=I+1}^{I_o} \theta_{0,i}^2$. Now set $\tau_1 = \frac{4(\alpha + a_K)K_1}{5}$. The last two relations and (29) imply that, for each $I \in \mathcal{N}^-$,

$$\begin{aligned} E_{\theta_0} p_I &= E_{\theta_0} P(\mathcal{I} = I|X) \leq e^{-(\alpha + a_K)I} \exp \left\{ (\alpha + a_K)I_o - \frac{1}{4} \sum_{i=1}^{I_o} \frac{\theta_{0,i}^2}{\sigma_i^2} \right\} \\ &\leq e^{-(\alpha + a_K)I} \exp \left\{ \frac{(\alpha + a_K)K_1}{\sigma_{I_o}^2} \sum_{i=1}^{I_o} \sigma_i^2 - \frac{1}{4\sigma_{I_o}^2} \sum_{i=I+1}^{I_o} \theta_{0,i}^2 \right\} \\ &\leq e^{-(\alpha + a_K)I} \exp \left\{ -\frac{1}{\sigma_{I_o}^2} \sum_{i=I+1}^{I_o} \theta_{0,i}^2 \right\}. \end{aligned} \quad (36)$$

Using the fact that $\max_{x \geq 0} \{x e^{-cx}\} \leq (ce)^{-1}$ (for any $c > 0$), (32), (33) and (36), we obtain

$$E_{\theta_0} \sum_{I \in \mathcal{N}^-} v_I p_I \leq E_{\theta_0} \sum_{I \in \mathcal{N}^-} \frac{r^2(\theta_0) + \sum_{i=I+1}^{I_o} \theta_{0,i}^2 + \sum_{i \leq I} \sigma_i^2 \xi_i^2}{M^2 r^2(\theta_0)} p_I$$

$$\begin{aligned}
&\leq \frac{1}{M^2} + \frac{\mathbb{E}_{\theta_0} \sum_{i \leq I_o} \sigma_i^2 \xi_i^2}{M^2 r^2(\theta_0)} + \sum_{I \in \mathcal{N}^-} \frac{(\sum_{i=I+1}^{I_o} \theta_{0,i}^2) \mathbb{E}_{\theta_0} p_I}{M^2 r^2(\theta_0)} \\
&\leq \frac{2}{M^2} + \sum_{I \in \mathcal{N}^-} \frac{(\sum_{i=I+1}^{I_o} \theta_{0,i}^2) \exp\{-\sigma_{I_o}^{-2} \sum_{i=I+1}^{I_o} \theta_{0,i}^2\} e^{-(\alpha+a_K)I}}{M^2 r^2(\theta_0)} \\
&\leq \frac{2}{M^2} + \sum_{I \in \mathcal{N}^-} \frac{e^{-(\alpha+a_K)I} e^{-1} \sigma_{I_o}^2}{M^2 r^2(\theta_0)} \leq \frac{2}{M^2} + \frac{e^{-1}}{M^2} \sum_I e^{-(\alpha+a_K)I} = \frac{C_1}{M^2}.
\end{aligned}$$

where $C_1 = 2 + \frac{e^{-(1+\alpha+a_K)}}{1-e^{-(\alpha+a_K)}}$, $\tau_1 = \frac{4(\alpha+a_K)K_1}{5}$. The last relation and (35) give

$$\mathbb{E}_{\theta_0} T_1 = \mathbb{E}_{\theta_0} \sum_{I \in \mathcal{O}^-(\tau_1, \theta_0)} v_I p_I + \mathbb{E}_{\theta_0} \sum_{I \in \mathcal{N}^-(\tau_1, \theta_0)} v_I p_I \leq \frac{C_2}{M^2}, \quad (37)$$

where $C_2 = 2 + \tau_1 + C_1 = 4 + \frac{4(\alpha+a_K)K_1}{5} + \frac{e^{-(1+\alpha+a_K)}}{1-e^{-(\alpha+a_K)}}$.

Step 4: handling the term T_2 Since $p_I \in [0, 1]$ and $\sum_I p_I = 1$, $\mathbb{E}_{\theta_0} T_2 = \mathbb{E}_{\theta_0} \sum_{I_o < I \leq \tau I_o} v_I p_I \leq \mathbb{E}_{\theta_0} [\max_{I_o < I \leq \tau I_o} v_I]$. Using this, (32), (33), (34) and the property (ii) from (10), we get

$$\begin{aligned}
\mathbb{E}_{\theta_0} T_2 &\leq \mathbb{E}_{\theta_0} \max_{I_o < I \leq \tau I_o} v_I \leq \frac{\max_{I_o < I \leq \tau I_o} r^2(I, \theta_0) + \mathbb{E}_{\theta_0} \sum_{i \leq \tau I_o} \sigma_i^2 \xi_i^2}{M^2 r^2(\theta_0)} \\
&\leq \frac{1}{M^2} + \frac{2 \sum_{i \leq \tau I_o} \sigma_i^2}{M^2 r^2(\theta_0)} \leq \frac{1}{M^2} + \frac{2K_2(\tau) \sum_{i=1}^{I_o} \sigma_i^2}{M^2 r^2(\theta_0)} \leq \frac{1 + 2K_2(\tau)}{M^2}.
\end{aligned} \quad (38)$$

Step 5: handling the term T_3 For some $\tau_2 > 0$ to be chosen later, introduce the sets

$$\begin{aligned}
\mathcal{O}^+ &= \mathcal{O}^+(\tau, \tau_2, \theta_0) = \left\{ I \in \mathbb{N} : I > \tau I_o, \sum_{i=I_o+1}^I \sigma_i^2 \leq \tau_2 \sum_{i>I_o} \theta_{0,i}^2 \right\}, \\
\mathcal{N}^+ &= \mathcal{N}^+(\tau, \tau_2, \theta_0) = \left\{ I \in \mathbb{N} : I > \tau I_o, \sum_{i=I_o+1}^I \sigma_i^2 > \tau_2 \sum_{i>I_o} \theta_{0,i}^2 \right\}.
\end{aligned}$$

By (32), $\max_{I \in \mathcal{O}^+} r^2(I, \theta_0) \leq (1 + \tau_2) r^2(\theta_0)$. Let $I^+ = \max\{\mathcal{O}^+\}$, then $\sum_{i \leq I^+} \sigma_i^2 \leq \sum_{i=1}^{I_o} \sigma_i^2 + \tau_2 \sum_{i>I_o} \theta_{0,i}^2 \leq (1 \vee \tau_2) r^2(\theta_0)$. In view of (33), the last two relations entail that

$$\mathbb{E}_{\theta_0} \sum_{I \in \mathcal{O}^+} v_I p_I \leq \mathbb{E}_{\theta_0} \max_{I \in \mathcal{O}^+} v_I \leq \frac{1 + \tau_2}{M^2} + \frac{\mathbb{E}_{\theta_0} \sum_{i \leq I^+} \sigma_i^2 \xi_i^2}{M^2 r^2(\theta_0)} \leq \frac{2(1 + \tau_2)}{M^2}. \quad (39)$$

Let K_4 and $\tau > 2$ be from property (iv) of (10), then $\Sigma(m) - \Sigma(\lfloor m/\tau \rfloor) \geq K_4 \Sigma(m)$ for any $m \geq \tau$. This entails that, for each $I \in \mathcal{N}^+$,

$$\sum_{i=\lfloor I/\tau \rfloor+1}^I \sigma_i^2 \geq K_4 \sum_{i=1}^I \sigma_i^2 \geq K_4 \sum_{i=I_o+1}^I \sigma_i^2 \geq K_4 \tau_2 \sum_{i=I_o+1}^I \theta_i^2 \geq K_4 \tau_2 \sum_{i=\lfloor I/\tau \rfloor+1}^I \theta_i^2.$$

For each $I \in \mathcal{N}^+$, take $I_0 = I_0(I) = \lfloor I/\tau \rfloor$, then apply the property (v) of (10) and the last inequality with $\tau_2 = \frac{10\tau}{9\alpha(\tau-2)K_4K_5(\tau)}$ to derive

$$\begin{aligned} I - 2I_0 - \frac{10}{9\alpha} \sum_{i=I_0+1}^I \frac{\theta_{0,i}^2}{\sigma_i^2} &\geq (1 - \frac{2}{\tau})I - \frac{10}{9\alpha} \sum_{i=I_0+1}^I \frac{\theta_{0,i}^2}{\sigma_i^2} \\ &\geq (1 - \frac{2}{\tau})K_5 \sum_{i=I_0+1}^I \frac{\sigma_i^2}{\sigma_{I_0}^2} - \frac{10}{9\alpha} \sum_{i=I_0+1}^I \frac{\theta_{0,i}^2}{\sigma_{I_0}^2} \\ &\geq \frac{(\tau-2)K_5(\tau)K_4\tau_2}{\tau} \sum_{i=I_0+1}^I \frac{\theta_{0,i}^2}{\sigma_{I_0}^2} - \frac{10}{9\alpha} \sum_{i=I_0+1}^I \frac{\theta_{0,i}^2}{\sigma_{I_0}^2} = 0. \end{aligned}$$

The last relation and the bound (31) with $I_0 = \lfloor I/\tau \rfloor$ imply that

$$\mathbb{E}_{\theta_0} p_I = \mathbb{E}_{\theta_0} \mathbb{P}(\mathcal{I} = I | X) \leq e^{-\gamma I}, \quad I \in \mathcal{N}^+, \quad \gamma = \frac{\alpha}{20}. \quad (40)$$

Since $p_I \in [0, 1]$ and $\mathbb{E} \left[\sum_{i=1}^m \sigma_i^2 \xi_i^2 \right]^2 \leq 3 \left[\sum_{i=1}^m \sigma_i^2 \right]^2$ for any $m \in \mathbb{N}$, we obtain by the Cauchy-Schwartz inequality that

$$\mathbb{E}_{\theta_0} \left[p_I \sum_{i \leq I} \sigma_i^2 \xi_i^2 \right] \leq (\mathbb{E}_{\theta_0} p_I^2)^{1/2} \sqrt{3} \sum_{i \leq I} \sigma_i^2 \leq \sqrt{3} (\mathbb{E}_{\theta_0} p_I)^{1/2} \sum_{i \leq I} \sigma_i^2. \quad (41)$$

Combining (32), (33), (40), (41), the property (iii) of (10) and the fact that $\varepsilon^2 = \sigma_1^2 \leq r^2(\theta_0)$, we derive

$$\begin{aligned} \mathbb{E}_{\theta_0} \sum_{I \in \mathcal{N}^+} p_I v_I &= \sum_{I \in \mathcal{N}^+(\tau, \tau_2)} \frac{r^2(I, \theta_0) \mathbb{E}_{\theta_0} p_I + \mathbb{E}_{\theta_0} [p_I \sum_{i \leq I} \sigma_i^2 \xi_i^2]}{M^2 r^2(\theta_0)} \\ &\leq \frac{1}{M^2} + \sum_{I \in \mathcal{N}^+} \frac{(\sum_{i=I_0+1}^I \sigma_i^2) \mathbb{E}_{\theta_0} p_I + \sqrt{3} (\sum_{i \leq I} \sigma_i^2) (\mathbb{E}_{\theta_0} p_I)^{1/2}}{M^2 r^2(\theta_0)} \\ &\leq \frac{1}{M^2} + \sum_{I \in \mathcal{N}^+} \frac{\varepsilon^2 [(\sum_{i=I_0+1}^I \kappa_i^2) e^{-\gamma I} + \sqrt{3} (\sum_{i \leq I} \kappa_i^2) e^{-\gamma I/2}]}{M^2 r^2(\theta_0)} \\ &\leq \frac{1 + K_3(\gamma) + \sqrt{3} K_3(\gamma/2)}{M^2}. \end{aligned}$$

Finally, the last relation and (39) entail the bound

$$\mathbb{E}_{\theta_0} T_3 = \mathbb{E}_{\theta_0} \sum_{I \in \mathcal{O}^+(\tau, \tau_2, \theta_0)} v_I p_I + \mathbb{E}_{\theta_0} \sum_{I \in \mathcal{N}^+(\tau, \tau_2, \theta_0)} v_I p_I \leq \frac{C_3}{M^2}, \quad (42)$$

where $C_3 = 2(1 + \tau_2) + 1 + K_3(\gamma) + \sqrt{3} K_3(\gamma/2)$.

Step 6: finalizing the proof Piecing together the relations (34), (37), (38) and (42), we finally obtain

$$\mathbb{E}_{\theta_0} \mathbb{P}(\|\theta - \theta_0\| \geq Mr(\theta_0)|X) \leq \mathbb{E}_{\theta_0}(T_1 + T_2 + T_3) \leq \frac{C_{or}}{M^2}.$$

The constant $C_{or} = C_{or}(K, \alpha)$ is as follows:

$$C_{or} = C_2 + 1 + 2K_2(\tau) + 2(1 + \tau_2) + 1 + K_3(\gamma) + \sqrt{3}K_3(\gamma/2),$$

where $C_2 = 4 + \frac{4(\alpha+a_K)K_1}{5} + \frac{e^{-(1+\alpha+a_K)}}{1-e^{-(\alpha+a_K)}}$, $a_K = \frac{1}{2} \log \left(\frac{K+1}{2} \right)$, $\tau_2 = \frac{10\tau}{9\alpha(\tau-2)K_4K_5(\tau)}$, $\gamma = \frac{\alpha}{20}$, the constants $\tau, K_1, K_2, K_3, K_4, K_5$ are from (10).

4.2 Proof of Theorem 3

Step 1: first technical lemma

Lemma 1. *Let $\text{DDM } \mathbb{P}_{K,\alpha}(\mathcal{I} = I|X)$ be given by (13) with parameters $K, \alpha > 0$ chosen in such a way that $a(K) > \alpha$, with $a(K)$ defined by (24). Let $\varkappa_0 = \varkappa_0(K, \alpha) = \frac{a(K)-\alpha}{a(K)}$. Then for any $\theta_0 \in \ell_2$ and any $\varkappa \in [0, \varkappa_0]$*

$$\mathbb{E}_{\theta_0} \mathbb{P}(\mathcal{I} \leq \varkappa \bar{I}_o | X) \leq C \exp \{ -c \bar{I}_o \}, \quad (43)$$

where $c = a(K)(1 - \varkappa) - \alpha > 0$, $C = C_\alpha^{-1} = (e^\alpha - 1)^{-1}$, and $\bar{I}_o = \bar{I}_o(\theta_0)$ is defined by (23).

Proof of Lemma 1. By the definition (23) of the surrogate oracle, $R^2(I, \theta_0) \geq R^2(\bar{I}_o, \theta_0)$ for any $\theta_0 \in \ell_2$. For $I < \bar{I}_o$, this implies that $\sum_{i=I+1}^{\bar{I}_o} \frac{\theta_{0,i}^2}{\sigma_i^2} \geq \bar{I}_o - I$. Using this, we obtain that for $I \leq \varkappa \bar{I}_o$

$$\begin{aligned} \frac{1}{4} \sum_{i=I+1}^{\bar{I}_o} \frac{\theta_{0,i}^2}{\sigma_i^2} - \frac{1}{2} \log \left[\frac{K+1}{2} \right] (\bar{I}_o - I) &\geq \left(\frac{1}{4} - \frac{1}{2} \log \left[\frac{K+1}{2} \right] \right) (\bar{I}_o - I) \\ &= a(K)(\bar{I}_o - I) \geq a(K)(1 - \varkappa) \bar{I}_o. \end{aligned}$$

The lemma follows from the last relation, (29) and the fact that $\sum_I \lambda_I = 1$:

$$\begin{aligned} \mathbb{E}_{\theta_0} \mathbb{P}(\mathcal{I} \leq \varkappa \bar{I}_o | X) &\leq \sum_{I \leq \varkappa \bar{I}_o} \frac{\lambda_I}{\lambda_{\bar{I}_o}} \exp \left\{ -\frac{1}{4} \sum_{i=I+1}^{\bar{I}_o} \frac{\theta_{0,i}^2}{\sigma_i^2} - \frac{1}{2} \log \left[\frac{K+1}{2} \right] (\bar{I}_o - I) \right\} \\ &\leq \sum_{I \leq \varkappa \bar{I}_o} \frac{\lambda_I}{\lambda_{\bar{I}_o}} \exp \{ -(a(K)(1 - \varkappa)) \bar{I}_o \} \leq \frac{1}{C_\alpha} \exp \{ -(a(K)(1 - \varkappa) - \alpha) \bar{I}_o \}. \quad \square \end{aligned}$$

Step 2: second technical lemma

Lemma 2. *Let $\Lambda(S)$ be the Lebesgue measure (or volume) of a bounded set $S \subset \mathbb{R}^k$, $k \in \mathbb{N}$, and $B_k(r) = \{x \in \mathbb{R}^k : \|x\| \leq r\}$ (here $\|\cdot\|$ is the usual Euclidean norm in \mathbb{R}^k) be the Euclidean ball of radius r in space \mathbb{R}^k . Then*

$$\Lambda(B_k(r)) \leq e\pi^{-1/2} r^k k^{-(k+1)/2} (2\pi e)^{k/2}.$$

Proof of Lemma 2. By using Stirling's approximation for the Gamma function $\Gamma(x) = \sqrt{2\pi} x^{x-1/2} e^{-x+\varsigma/(12x)}$ for all $x \geq 1$ and some $0 \leq \varsigma \leq C$, we derive

$$\begin{aligned}\Gamma\left(1 + \frac{k}{2}\right) &= \sqrt{2\pi} \left(1 + \frac{k}{2}\right)^{\frac{k+1}{2}} e^{-1-\frac{k}{2}+\frac{\varsigma}{6k+12}} = \frac{\left(1 + \frac{2}{k}\right)^{(k+1)/2} \sqrt{\pi}}{e^{1-\varsigma/(6k+12)}} k^{\frac{k+1}{2}} (2e)^{-\frac{k}{2}} \\ &= c_k k^{(k+1)/2} (2e)^{-k/2} \geq e^{-1} \pi^{1/2} k^{(k+1)/2} (2e)^{-k/2},\end{aligned}$$

because $c_k = \frac{(1+2/k)^{(k+1)/2} \sqrt{\pi}}{e^{1-\varsigma/(6k+12)}} > \frac{\sqrt{\pi}}{e}$. Combining the last relation with the well known fact that $\Lambda(B_k(r)) = r^k \Lambda(B_k(1)) = \frac{r^k \pi^{k/2}}{\Gamma(1+k/2)}$ completes the proof of the lemma. \square

Step 3: small ball bound for $P_I(\cdot|X)$ Recall that, with $L = K/(K+1)$,

$$P_I(\theta|X) = \bigotimes_i N(X_i 1\{i \leq I\}, L\sigma_i^2 1\{i \leq I\}), \quad I \in \mathbb{N}.$$

We have that $\Sigma(I) = \sum_{i=1}^I \sigma_i^2 \leq \varepsilon^2 \frac{(2I)^{2p+1}}{2p+1}$. By Stirling's bound, $\prod_{i=1}^I \kappa_i = (I!)^p \geq ((I/e)^I \sqrt{2\pi I})^p$. Let Z_1, \dots, Z_I be independent $N(0, 1)$ random variables. Using these relations, Anderson's inequality and Lemma 2, we obtain that, P_{θ_0} -almost surely,

$$\begin{aligned}P_I(\|\theta - \hat{\theta}\| \leq \delta \Sigma^{1/2}(\bar{I}_o)|X) &= P_I(\|\theta - \hat{\theta}\|^2 \leq \delta^2 \Sigma(\bar{I}_o)|X) \\ &= P\left(\sum_{i \leq I} (X_i + \sigma_i \sqrt{L} Z_i - \hat{\theta}_i)^2 + \sum_{i > I} \hat{\theta}_i^2 \leq \delta^2 \Sigma(\bar{I}_o)|X\right) \\ &\leq P\left(L \sum_{i \leq I} \sigma_i^2 Z_i^2 \leq \delta^2 \Sigma(\bar{I}_o)\right) \leq P\left(\sum_{i \leq I} \sigma_i^2 Z_i^2 \leq \frac{\delta^2 \Sigma(\bar{I}_o)}{L}\right) \\ &\leq \frac{\Lambda(B_I(\delta \sqrt{\Sigma(\bar{I}_o)/L}))}{\prod_{i=1}^I (2\pi \sigma_i^2)^{1/2}} \leq \frac{(2\pi)^{-I/2}}{\prod_{i=1}^I \varepsilon \kappa_i} \frac{e}{\sqrt{\pi}} \left(\frac{\delta^2 \Sigma(\bar{I}_o)}{L}\right)^{I/2} I^{-\frac{I+1}{2}} (2\pi e)^{I/2} \\ &\leq \frac{e I^{-(p+1)/2}}{(2\pi)^{p/2} \sqrt{\pi}} \left[\left(\frac{2e \bar{I}_o}{I}\right)^{p+1/2} \left(\frac{\delta}{\sqrt{L(2p+1)}}\right)\right]^I.\end{aligned}\tag{44}$$

Step 4: applying Lemma 1 Denote for brevity $\varrho = a(K) - \alpha$. By (24), $\varrho > 0$. Applying Lemma 1 with $\varkappa = \frac{\varkappa_0}{2} = \frac{a(K) - \alpha}{2a(K)}$ (so that $a(K)(1 - \varkappa) - \alpha = \frac{a(K) - \alpha}{2} = \frac{\varrho}{2}$), we obtain

$$E_{\theta_0} P(\mathcal{I} < \varkappa \bar{I}_o | X) \leq C_\alpha^{-1} e^{-\varrho \bar{I}_o/2} \tag{45}$$

for every $\theta_0 \in \ell_2$. Consider the two cases: $e^{-\varrho \bar{I}_o/2} \leq \delta$ and $e^{-\varrho \bar{I}_o/2} > \delta$.

Step 5: the case $e^{-\varrho \bar{I}_o/2} > \delta$ If $e^{-\varrho \bar{I}_o/2} > \delta$, then $\bar{I}_o < 2\varrho^{-1} \log(\delta^{-1})$. By using this, (11), (44) and the notation $p_I = P(\mathcal{I} = I|X)$, we derive that, for $e^{-\varrho \bar{I}_o/2} > \delta$,

$$E_{\theta_0} P(\|\theta - \hat{\theta}\| \leq \delta \Sigma^{1/2}(\bar{I}_o)|X) = E_{\theta_0} \sum_I P_I(\|\theta - \hat{\theta}\|^2 \leq \delta^2 \Sigma(\bar{I}_o)|X) p_I$$

$$\begin{aligned}
&\leq \sum_I \frac{e^{I-(p+1)/2}}{(2\pi)^{p/2}\sqrt{\pi}} \left[\left(\frac{2e\bar{I}_o}{I} \right)^{p+1/2} \left(\frac{\delta}{\sqrt{L(2p+1)}} \right) \right]^I \mathbb{E}_{\theta_0} p_I \\
&\leq C_2 \delta [\log(\delta^{-1})]^{p+1/2} \sum_I \frac{(C_1 \delta [\log(\delta^{-1})]^{p+1/2})^{I-1}}{I^{I(p+1/2)+(p+1)/2}} \mathbb{E}_{\theta_0} p_I \\
&\leq C_3 \delta [\log(\delta^{-1})]^{p+1/2}, \tag{46}
\end{aligned}$$

with $C_1 = \frac{(4e/\varrho)^{p+1/2}}{(L(2p+1))^{1/2}}$, $C_2 = \frac{C_1 e}{\pi^{1/2}(2\pi)^{p/2}}$, $\varrho = a(K) - \alpha$, $L = \frac{K}{K+1}$, and some $C_3 = C_3(\varrho, L, p)$.

Step 6: the case $e^{-\varrho\bar{I}_o/2} \leq \delta$ Now consider the case $e^{-\varrho\bar{I}_o/2} \leq \delta$. Clearly, $\sum_{I < \varkappa\bar{I}_o} p_I = \mathbb{P}(\mathcal{I} < \varkappa\bar{I}_o | X)$. In view of this, (44) and (45),

$$\begin{aligned}
&\mathbb{E}_{\theta_0} \mathbb{P}(\|\theta - \hat{\theta}\| \leq \delta \Sigma^{1/2}(\bar{I}_o) | X) = \mathbb{E}_{\theta_0} \sum_I \mathbb{P}_I(\|\theta - \hat{\theta}\|^2 \leq \delta^2 \Sigma(\bar{I}_o) | X) p_I \\
&\leq \sum_{I \geq \varkappa\bar{I}_o} \frac{e^{I-\frac{p+1}{2}}}{(2\pi)^{\frac{p}{2}}\sqrt{\pi}} \left[\left(\frac{2e\bar{I}_o}{I} \right)^{p+\frac{1}{2}} \left(\frac{\delta}{\sqrt{L(2p+1)}} \right) \right]^I \mathbb{E}_{\theta_0} p_I + \mathbb{E}_{\theta_0} \mathbb{P}(\mathcal{I} < \varkappa\bar{I}_o | X) \\
&\leq C_4 \delta \sum_I \frac{\left[\left(\frac{2e}{\varkappa} \right)^{p+\frac{1}{2}} \frac{\delta}{\sqrt{L(2p+1)}} \right]^{I-1}}{I^{(p+1)/2}} \mathbb{E}_{\theta_0} p_I + \frac{e^{-\varrho\bar{I}_o/2}}{C_\alpha} \leq (C_4 + C_\alpha^{-1})\delta
\end{aligned}$$

if $e^{-\varrho\bar{I}_o/2} \leq \delta$ and $\left(\frac{2e}{\varkappa} \right)^{p+\frac{1}{2}} \frac{\delta}{\sqrt{L(2p+1)}} \leq 1$. Here $C_4 = \frac{e}{(2\pi)^{\frac{p}{2}}\sqrt{\pi L(2p+1)}} \left(\frac{2e}{\varkappa} \right)^{p+\frac{1}{2}}$.

Step 7: finalizing the proof of Theorem 3 The last relation holds if $e^{-\varrho\bar{I}_o/2} \leq \delta \leq \sqrt{L(2p+1)} \left(\frac{\varkappa}{2e} \right)^{p+1/2} = \sqrt{\frac{K(2p+1)}{K+1}} \left(\frac{\varkappa}{2e} \right)^{p+1/2} = \bar{\delta}_{sb}$ and the relation (46) holds if $e^{-\varrho\bar{I}_o/2} > \delta$. Combining these two relations concludes the proof of the theorem: for $0 < \delta \leq (1 \wedge \bar{\delta}_{sb}) = \delta_{sb}$, we have that

$$\mathbb{E}_{\theta_0} \mathbb{P}(\|\theta - \hat{\theta}\| \leq \delta \Sigma^{1/2}(\bar{I}_o) | X) \leq \max\{C_3, C_4 + C_\alpha^{-1}\} \delta [\log(\delta^{-1})]^{p+\frac{1}{2}}.$$

References

- [1] BARAUD, Y. (2004). Confidence balls in Gaussian regression. *Ann. Statist.* 32, 528–551.
- [2] BELITSER, E. (2015). Supplement to “On coverage and local radial rates of DDM-credible sets”.
- [3] BERAN, R. and DÜMBGEN, L. (1998). Modulation of estimators and confidence sets. *Ann. Statist.* 26, 1826–1856.
- [4] BIRGÉ, L. and MASSART, P. (2001). Gaussian model selection. *J. Eur. Math. Soc.* 3 203–268.

- [5] BULL, A. (2012). Honest adaptive confidence bands and self-similar functions. *Electron. J. Statist.* 6, 1490–1516.
- [6] BULL, A. and NICKL, R. (2013). Adaptive confidence sets in ℓ^2 . *Probab. Theory and Rel. Fields* 156, 889–919.
- [7] CAI, T.T. and LOW, M.G. (2006). Adaptive confidence balls. *Ann. Statist.* 34, 202–228.
- [8] CAVALIER, L. (2008). Nonparametric statistical inverse problems. *Inverse Problems* 24, 1–19.
- [9] CAVALIER, L. and GOLUBEV, YU. (2006). Risk hull method and regularization by projections of ill-posed inverse problems. *Ann. Statist.* 34, 1653–1677.
- [10] GENOVESE, C.R. and WASSERMAN, L. (2008). Adaptive confidence bands. *Ann. Statist.* 36, 875–905.
- [11] GINÉ, E. and NICKL, R. (2010). Confidence bands in density estimation. *Ann. Statist.* 38, 1122–1170.
- [12] GHOSAL, S., GHOSH, J. and VAN DER VAART, A.W. (2000). Convergence rates of posterior distributions. *Ann. Statist.* 28, 500–531.
- [13] JUDITSKY, A. and LAMBERT-LACROIX, S. (2003). Nonparametric confidence set estimation. *Math. Meth. Statist.* 12, 410–428.
- [14] HOFFMANN, M. and NICKL, R. (2011). On adaptive inference and confidence bands. *Ann. Statist.* 39, 2382–2409.
- [15] LI, K.-C. (1989). Honest confidence regions for nonparametric regression. *Ann. Statist.* 17, 1001–1008.
- [16] LOW, M.G. (1997). On nonparametric confidence intervals. *Ann. Statist.* 25, 2547–2554.
- [17] NICKL, R. and SZABÓ, B.T. (2014). A sharp adaptive confidence ball for self-similar functions. arXiv:1406.3994.
- [18] PICARD, D. and TRIBOULEY, K. (2000). Adaptive confidence interval for pointwise curve estimation. *Ann. Statist.* 28, 298–335.
- [19] ROBINS, J. and VAN DER VAART, A.W. (2006). Adaptive nonparametric confidence sets. *Ann. Statist.* 34, 229–253.
- [20] SERRA, P. and KRIVIBOKOVA, T. (2014). Adaptive empirical Bayesian smoothing splines. arXiv:1411.6860.
- [21] SZABÓ, B.T., VAN DER VAART, A.W. and VAN ZANTEN, J.H. (2015). Frequentist coverage of adaptive nonparametric Bayesian credible sets. *Ann. Statist.* 43, 1391–1428.
- [22] SZABÓ, B.T., VAN DER VAART, A.W. and VAN ZANTEN, J.H. (2015b). Honest Bayesian confidence sets for the ℓ^2 -norm. To appear in *J. Stat. Plan. Inference*.

A Supplement to “On coverage and local radial rates of DDM-credible sets”

In this supplement, we provide the elaboration on some points and some background information related to the paper “On coverage and local radial rates of DDM-credible sets”.

In what follows we use the notations and cross-references to numbered elements (like equations, sections) from the paper. We again often drop the dependence on ε to avoid overloaded notations. For two sequences $\alpha_\varepsilon, \beta_\varepsilon > 0$, $\alpha_\varepsilon \asymp \beta_\varepsilon$ means that $\alpha_\varepsilon/\beta_\varepsilon$ is bounded away from zero and infinity as $\varepsilon \rightarrow 0$.

A.1 Minimax confidence ball: degenerate optimal solution

Optimality is a well developed notion in the framework of minimax estimation theory and therefore the first approach to optimality of confidence sets would be based on the minimax convergence rates. Suppose our prior knowledge about the model $X \sim P_\theta = P_\theta^{(\varepsilon)}$ is formalized as follows: $\theta \in \Theta_\beta \subseteq \Theta$. Here we consider *non-adaptive* situation, that is, the parameter $\beta \in \mathcal{B}$ is known and we can use this knowledge in the construction of the confidence ball. Parameter β typically has a meaning of smoothness of θ . By using lower bounds from the minimax estimation theory, we show below that the minimax rate $R_\varepsilon(\Theta_\beta)$ is in some sense the *best global radial rate*, i.e., the smallest possible among all radial rates that are constant on Θ_β .

Let $w : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, be a loss function, i.e., nonnegative and nondecreasing on \mathbb{R}_+ , $w(0) = 0$ and $w \not\equiv 0$. The *maximal risk* of an estimator $\hat{\theta}$ over Θ_β is $r_\varepsilon(\Theta_\beta, \hat{\theta}) = r_\varepsilon(\Theta_\beta, \hat{\theta}, R_\varepsilon) = \sup_{\theta \in \Theta_\beta} E_\theta[w(R_\varepsilon^{-1}d(\hat{\theta}, \theta))]$ (calibrated by a sequence $R_\varepsilon > 0$), and the *minimax risk* over Θ_β is $r_\varepsilon(\Theta_\beta) = r_\varepsilon(\Theta_\beta, R_\varepsilon) = \inf_{\hat{\theta}} r_\varepsilon(\Theta_\beta, \hat{\theta}, R_\varepsilon)$, where the infimum is taken over all possible estimators $\hat{\theta} = \hat{\theta}(X) \in \mathcal{L}$, measurable functions of the data X . We consider here the asymptotic regime $\varepsilon \rightarrow 0$ as in the most literature on minimax estimation theory. A positive sequence $R_\varepsilon = R_\varepsilon(\Theta_\beta)$ and an estimator $\hat{\theta}$ are called *minimax rate* and *minimax estimator* respectively if, for $0 < b \leq B < \infty$,

$$b \leq \liminf_{\varepsilon \rightarrow 0} r_\varepsilon(\Theta_\beta, R_\varepsilon) \leq \limsup_{\varepsilon \rightarrow 0} r_\varepsilon(\Theta_\beta, \hat{\theta}, R_\varepsilon) \leq B. \quad (\text{S1})$$

The first inequality is called lower bound and the last one upper bound. Note that the minimax rate is not unique. If $w(u) = u^p$, $p > 0$ (the most popular choice: quadratic loss function $p = 2$), then often the quantity $r_\varepsilon(\Theta_\beta) = \inf_{\hat{\theta}} \sup_{\theta \in \Theta_\beta} (E_\theta[d(\hat{\theta}, \theta)]^p)^{1/p}$, is called the minimax risk. In this case, the minimax risk is itself the minimax rate, but so is any sequence $R_\varepsilon(\Theta_\beta) \asymp r_\varepsilon(\Theta_\beta)$. If the set Θ_β is known, $r_\varepsilon(\Theta_\beta)$ is in principle known as well, one would like to derive an explicit expression $R_\varepsilon(\Theta_\beta)$ for the minimax rate. There is vast literature on this topic, minimax rates and estimators are obtained in a variety of models, settings and smoothness classes Θ_β . For example, in classical nonparametric regression model and density estimation problem with Sobolev, Hölder or Besov classes Θ_β of d -variate functions of smoothness β and the sample size n , the minimax rate is $R_\varepsilon(\Theta_\beta) = (\varepsilon^2)^{\frac{\beta}{2\beta+d}}$ with $\varepsilon = n^{-1/2}$.

Suppose that a lower bound in (S1) is established for zero-one loss $w(u) = 1\{u \geq c\}$ and a (minimax) rate $R_\varepsilon(\Theta_\beta)$: for any $\hat{\theta}$ and some $b > 0$,

$$\liminf_{\varepsilon \rightarrow 0} \sup_{\theta \in \Theta_\beta} \mathbb{P}_\theta(\theta \notin B(\hat{\theta}, cR_\varepsilon(\Theta_\beta))) = \liminf_{\varepsilon \rightarrow 0} r_\varepsilon(\Theta_\beta, \hat{\theta}, R_\varepsilon(\Theta_\beta)) \geq b. \quad (\text{S2})$$

We claim that it is impossible for a confidence ball $B(\hat{\theta}, \hat{r})$ to have simultaneously a global radial rate of a smaller order than $R_\varepsilon(\Theta_\beta)$ and its coverage probability being arbitrarily close to 1 uniformly in $\theta \in \Theta_\beta$.

There are two ways to establish lower bounds for the optimality of confidence sets: either assume the coverage relation in (1) and show that the size relation must fail or the other way around. In the literature, the former approach is commonly used for global minimax radial rates, cf. [10]. However, when we construct confidence sets as credible balls with respect to some DDM $\mathbb{P}(\cdot|X)$, it is more natural to use the latter approach since the DD-radius gets determined by the DDM and typically the size requirement in (1) holds true for the whole set Θ , whereas the coverage requirement fails to hold for some “deceptive” $\theta \in \Theta$.

More precisely, if we assume

$$\liminf_{\varepsilon \rightarrow 0} \inf_{\theta \in \Theta} \mathbb{P}_\theta(\hat{r} \leq cR_\varepsilon(\Theta_\beta)) \geq 1 - b/2, \quad (\text{S3})$$

then

$$\begin{aligned} \mathbb{P}_\theta(\theta \notin B(\hat{\theta}, \hat{r})) &= \mathbb{P}_\theta(\theta \notin B(\hat{\theta}, \hat{r}), \hat{r} \leq R_\varepsilon(\Theta_\beta)) + \mathbb{P}_\theta(\theta \notin B(\hat{\theta}, \hat{r}), \hat{r} > R_\varepsilon(\Theta_\beta)) \\ &\geq \mathbb{P}_\theta(\theta \notin B(\hat{\theta}, R_\varepsilon(\Theta_\beta)), \hat{r} \leq R_\varepsilon(\Theta_\beta)) \\ &\geq \mathbb{P}_\theta(\theta \notin B(\hat{\theta}, R_\varepsilon(\Theta_\beta))) + \mathbb{P}_\theta(\hat{r} \leq R_\varepsilon(\Theta_\beta)) - 1. \end{aligned}$$

Combining this with (S2) and (S3), we obtain

$$\liminf_{\varepsilon \rightarrow 0} \sup_{\theta \in \Theta_\beta} \mathbb{P}_\theta(\theta \notin B(\hat{\theta}, \hat{r})) \geq b + 1 - b/2 - 1 \geq b/2,$$

which gives a bound on the coverage probability of $B(\hat{\theta}, \hat{r})$, at least for some (worst representatives) $\theta \in \Theta_\beta$. We thus established that it is impossible for a confidence ball $B(\hat{\theta}, \hat{r})$ to have simultaneously a global radial rate of a smaller order than $R_\varepsilon(\Theta_\beta)$ and its coverage probability being arbitrarily close to 1 uniformly in $\theta \in \Theta_\beta$.

On the other hand, suppose now that there is a minimax estimator $\hat{\theta}$ satisfying (S1), with, say, $w(u) = u$, and the corresponding minimax rate $R_\varepsilon(\Theta_\beta)$. If we use the minimax risk $R_\varepsilon(\Theta_\beta)$ as the benchmark for the effective radius of confidence balls, then the problem of constructing an optimal confidence ball satisfying (1) with the radial rate $r_\varepsilon(\theta) = R_\varepsilon(\Theta_\beta)$ is readily solved. Indeed, since in this non-adaptive setting the quantity $R_\varepsilon(\Theta_\beta)$ is in principle known (could be difficult to evaluate in models), we can simply take the following confidence ball $B(\hat{\theta}, CR_\varepsilon(\Theta_\beta))$, i.e., $\hat{r} = R_\varepsilon(\Theta_\beta)$. Then, by (S1),

$$\limsup_{\varepsilon \rightarrow 0} \sup_{\theta \in \Theta_\beta} \mathbb{P}_\theta(\theta \notin B(\hat{\theta}, CR_\varepsilon(\Theta_\beta))) \leq \limsup_{\varepsilon \rightarrow 0} \frac{\sup_{\theta \in \Theta_\beta} \mathbb{E}_\theta d(\theta, \hat{\theta})}{CR_\varepsilon(\Theta_\beta)} \leq \frac{B}{C},$$

so that the coverage relation in (1) will hold for sufficiently large C . The size relation in (1) is trivially satisfied for any $c > 1$ any $\alpha_2 \in [0, 1]$ since $\hat{r} = r_\varepsilon(\theta) = R_\varepsilon(\Theta_\beta)$. This means that the ball $B(\hat{\theta}, CR_\varepsilon(\Theta_\beta))$ satisfies (1) with $r_\varepsilon(\theta) = R_\varepsilon(\Theta_\beta)$ and $\Theta_{cov} = \Theta_{size} = \Theta_\beta$, for appropriate choices of involved constants. Thus the ball $B(\hat{\theta}, CR_\varepsilon(\Theta_\beta))$, with the deterministic radius $R_\varepsilon(\Theta_\beta)$, is optimal in the minimax sense (in the non-adaptive formulation). Knowledge $\theta \in \Theta_\beta$ and the fact that radial rates are restricted to be global lead to such a simplistic optimal solution. But this solution is of course not satisfactory, because even if we know a priori that $\theta \in \Theta_\beta$, it is possible that $\theta \in \Theta_{\beta_1} \subset \Theta_\beta$, with $\beta_1 \neq \beta$. Then the obtained radial rate $R_\varepsilon(\Theta_\beta) > R_\varepsilon(\Theta_{\beta_1})$ is bigger than it could have been if one had used a ball with a DD-radius that can adapt to the rate $R_\varepsilon(\Theta_{\beta_1})$.

This consideration illustrates that minimax non-adaptive framework for the confidence inference lead to degenerate and uninteresting “optimal” solution.

A.2 Bayes approach yields DDMs

Suppose we are given a general statistical model $X \sim P_\theta$, $\theta \in \Theta$, and we want to construct a DDM on parameter θ . Typically, one obtains a DDM on θ by applying a Bayesian approach: put a prior π on θ and regard P_θ as conditional distribution of X given θ , i.e., $X|\theta \sim P_\theta$, $\theta \sim \pi$. This leads to the posterior distribution $\Pi(\theta|X)$ which is a DDM on θ . A DD-center $\hat{\theta} = \hat{\theta}(X)$ can in turn be constructed by using $\Pi(\theta|X)$, e.g., as the mean with respect to $\Pi(\theta|X)$ or the MAP-estimator. Other examples of DDMs include empirical Bayes, (generalized) fiducial distributions and bootstrap. In fact, any combination of these can be used as DDM.

In an adaptive inference context, one typically has a family of priors $\{\pi_\beta, \beta \in \mathcal{B}\}$, where parameter β models some additional structure on θ ; sometimes β has a meaning of “smoothness”. There are two basic approaches to derive a resulting adaptive posterior $\Pi(\theta|X)$: pure Bayes or empirical Bayes. In the first case, we construct a hierarchical prior on (θ, β) : regard π_β as a conditional prior on θ given β , and next we put a prior, say λ , on $\beta \in \mathcal{B}$. This leads to the posteriors $\Pi(\theta|X)$ and $\lambda(\beta|X)$ (that may be also useful in the inference). In the empirical Bayes approach, each prior π_β leads to the posterior $\Pi_\beta(\theta|X)$. We then compute the marginal distribution Π_β of X and construct an estimator $\hat{\beta}$ by using this marginal distribution (for example, marginal maximum likelihood). Next we plug in the obtained $\hat{\beta}$ in the posterior $\Pi_\beta(\theta|X)$, so that we get the so called empirical Bayes posterior $\hat{\Pi}(\theta|X) = \Pi_{\hat{\beta}}(\theta|X)$. Both resulting DDMs $\Pi(\theta|X)$ and $\hat{\Pi}(\theta|X)$ can be used in the construction of confidence sets as DDM-credible sets. Also any combination of full Bayes and empirical Bayes approaches (with respect to different parameters) that leads to some resulting DDM $P(\theta|X)$ can in principle be used.

To some extent, we can manipulate with DDMs as with usual conditional measures. For example, if we have a family of DDMs on Θ , say, $\{P_I(\cdot|X), I \in \mathbb{N}\}$ and a DDM $P(\mathcal{I} = I|X)$ on \mathbb{N} , we can construct a mixture DDM $P(\cdot|X) = \sum_I P_I(\cdot|X)P(\mathcal{I} = I|X)$.

A.3 Remarks about Conditions (A1)–A(3), ($\tilde{A}1$)–($\tilde{A}2$)

Here we collect some remarks about Conditions (A1)–A(3), ($\tilde{A}1$)–($\tilde{A}2$).

Asymptotic versions of conditions (A1)–(A3) and ($\tilde{A}1$)–($\tilde{A}2$) Suppose a point $\theta_0 \in \Theta$, some radial rate $r(\theta)$, a DDM $P(\cdot|X)$ and a DD-center $\hat{\theta} = \hat{\theta}(X)$ are given, $M_\varepsilon, M'_\varepsilon, \delta_\varepsilon > 0$ and $\varepsilon \rightarrow 0$. The asymptotic versions of conditions (A1)–(A3), ($\tilde{A}1$)–($\tilde{A}2$) are as follows.

(AA1) For some $M_\varepsilon \rightarrow \infty$, $E_{\theta_0}[P(d(\theta, \hat{\theta}) \geq M_\varepsilon r(\theta_0)|X)] \rightarrow 0$.

(AA2) For some $\delta_\varepsilon \rightarrow 0$, $E_{\theta_0}[P(d(\theta, \hat{\theta}) \leq \delta_\varepsilon r(\theta_0)|X)] \rightarrow 0$.

(AA3) For some $M'_\varepsilon \rightarrow \infty$, $P_{\theta_0}(d(\theta_0, \hat{\theta}) \geq M'_\varepsilon r(\theta_0)) \rightarrow 0$.

(A \tilde{A} 1) For some $M_\varepsilon \rightarrow \infty$, $E_{\theta_0}[P(d(\theta_0, \theta) \geq M_\varepsilon r(\theta_0)|X)] \rightarrow 0$.

(A \tilde{A} 2) For some $\delta_\varepsilon \rightarrow 0$ and any measurable $\tilde{\theta} = \tilde{\theta}(X)$, $E_{\theta_0}[P(d(\theta, \tilde{\theta}) \leq \delta_\varepsilon r(\theta_0)|X)] \rightarrow 0$.

Connection to Bayesian nonparametrics In the Bayesian framework, when the DDM $P(\cdot|X)$ is the posterior (or empirical Bayes posterior) distribution on θ with respect to some prior, condition ($\tilde{A}1$) (and its asymptotic version (A \tilde{A} 1) below) describes the so called posterior contraction rate $r(\theta_0)$. To establish such assertions is an interesting and challenging problem nowadays, especially in nonparametric models when one wants to characterize the (frequentist) quality of Bayesian procedures. Much recent research has been devoted to this topic. We just mention that predominantly global posterior convergence rates are studied, i.e., $r(\theta_0) = R(\Theta)$ for all $\theta_0 \in \Theta$. To the best of our knowledge a local posterior convergence rate is considered only in [1].

Pushing the conditions to the utmost The smaller the radial rate $r(\theta_0)$, the easier (A2) to satisfy, but the harder (A1), (A3) and ($\tilde{A}1$). We are interested in the smallest possible radial rate since this quantity will govern the size of the resulting confidence ball. Thus, the right strategy would be first to determine the smallest radial rate $r(\theta_0)$ for which ($\tilde{A}1$) (or (A1) and (A3)) holds, preferably uniformly over $\theta_0 \in \Theta$. This would be the so called upper bound for the contraction rate of the DDM $P(\cdot|X)$ around $\theta_0 \in \Theta$. Next, one needs to study whether (A2) holds as well with $r(\theta_0)$ for $\theta_0 \in \Theta$; if not possible for all $\theta_0 \in \Theta$, then for $\theta_0 \in \Theta_0$ with the “largest” $\Theta_0 \subset \Theta$. This is so called lower bound for the contraction rate of the DDM $P(\cdot|X)$ around $\hat{\theta}$.

Typically, the upper bound ($\tilde{A}1$) for the DDM-contraction rate holds for all $\theta \in \Theta$ with a “good” local radial rate, whereas the lower bound (A2) only for $\theta \in \Theta_0$, with some set of “non-deceptive” parameters $\Theta_0 \subset \Theta$.

A.4 Examples of applying Propositions 2 and 3

Normal case Suppose we observe a sample $X = X^{(\varepsilon)} = (X_1, \dots, X_n)$ from $N(\theta_0, \sigma^2)$, $\theta_0 \in \mathbb{R}$, where $\varepsilon = \sigma n^{-1/2}$. Take the estimator $\hat{\theta} = \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \sim N(\theta_0, \varepsilon^2)$ and the radial rate $r(\theta_0) = r_\varepsilon(\theta_0) = \varepsilon$. The normal prior $\pi = N(\mu, \tau^2)$ on θ , leads to the normal posterior $\pi(\theta|X) = N(\frac{\varepsilon^2 \mu + \tau^2 \bar{X}}{\varepsilon^2 + \tau^2}, \frac{\varepsilon^2 \tau^2}{\varepsilon^2 + \tau^2})$. Then, as DDM on θ we take

$$P(\theta|X) = \pi(\theta|X)|_{\mu=\hat{\mu}} = N\left(\bar{X}, \frac{\varepsilon^2 \tau^2}{\varepsilon^2 + \tau^2}\right),$$

the empirical Bayes posterior with $\hat{\mu} = \bar{X}$, and construct the DDM-credible ball (in this case: interval) $B(\hat{\theta}, M\hat{r}_\kappa)$ for θ_0 , according to the general procedure from the paper. Then (A1) and (A3) are satisfied with $\phi_1(M) = \phi_2(M) = Ce^{-cM^2}/M$. Indeed, for a $\xi \sim N(0, 1)$,

$$P(|\hat{\theta} - \theta| \geq Mr(\theta_0)|X) = P\left(\frac{\varepsilon\tau|\xi|}{\sqrt{\varepsilon^2 + \tau^2}} \geq M\varepsilon\right) \leq P(|\xi| \geq M) \leq \frac{2e^{-M^2/2}}{\sqrt{2\pi}M},$$

$$P(|\hat{\theta} - \theta_0| \geq Mr(\theta_0)) = P(|\varepsilon\xi| \geq M\varepsilon) = P(|\xi| \geq M) \leq \frac{2e^{-M^2/2}}{\sqrt{2\pi}M}.$$

Assume $\varepsilon \leq \tau$, then condition (A2) is also satisfied with $\psi(\delta) = \delta/\sqrt{\pi}$:

$$P(|\hat{\theta} - \theta| \leq \delta r(\theta_0)|X) = P(|\xi| \leq \delta\sqrt{1 + \varepsilon^2/\tau^2}) \leq P(|\xi| \leq \delta\sqrt{2}) \leq \delta/\sqrt{\pi}.$$

One can think of the above two properties of the normal distribution as “ring tightness”. The functions ϕ_1, ϕ_2 and ψ do not depend on ε and θ_0 , so that, by using Propositions 2 and 3 as described above, we can derive non-asymptotic coverage and size relations in (1) for the DDM-credible interval $B(\hat{\theta}, M\hat{r}_\kappa)$.

Of course, the classical confidence interval $\bar{X}_n \pm z_{1-\alpha/2}\sigma/\sqrt{n}$ has the same radial rate whose coverage may even be (non-asymptotically) better. In that respect, the above example is somewhat uninteresting and is provided only for the illustrative purposes.

Bernstein-von Mises case For the finite dimensional parameter, consider a general situation when some mild regularity conditions on the model and the prior lead to the resulting asymptotically normal posterior. This is the so called *Bernstein-von Mises* property as often termed in the literature. Suppose $X = X^{(n)} \sim P_{\theta_0}$, $\theta \in \Theta$, information parameter $\varepsilon = n^{-1/2}$, with a prior $\theta \sim \pi$ on some σ -algebra \mathcal{B}_Θ on Θ and a \sqrt{n} -consistent estimator $\hat{\theta}$ such that the asymptotic version of (A3), namely (AA3) (given in Subsection A.3), is satisfied with the radial rate $\mathcal{R}_n(\theta_0) = n^{-1/2}$ and in P_{θ_0} -probability

$$\sup_{B \in \mathcal{B}_\Theta} |\pi(B|X) - N(\hat{\theta}, I(\theta_0))(B)| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

where $N(\mu, \Sigma)(B) = P(Y \in B)$ with $B \sim N(\mu, \Sigma)$ for some multivariate normal distribution with mean μ and covariance matrix Σ . Besides, (A1) and (A2) hold for the DDM $N(\hat{\theta}, I(\theta_0))$. All these facts imply that the asymptotic versions (AA1)–(AA3) introduced in Supplement are satisfied with $\varepsilon = n^{-1/2}$. Asymptotic versions of Propositions 2 and 3 follow immediately, which yields (asymptotically) a full coverage probability and the optimal global radial rate $n^{-1/2}$, which is of course well known.

Interestingly, there is nothing special about normal distribution in the above arguments, any resulting limiting distribution with a “ring structure” will do the same job. Ring structure means negligible probability mass outside a ring, whose inner radius is a sufficiently small multiples of the radial rate and the outer radius is a sufficiently big multiples of the radial rate. In fact, the existence of an exact limiting distribution is also not decisive, “ring tightness” (which is nothing else but (AA1)–(AA2)) would be enough. For example, the Bernstein-von Mises property is more than needed if we only want to make sure that a credible set serves as a proper confidence set.

A.5 Corollary from Propositions 1–3

Propositions 1–3 and Remark 1 entail the following corollary for the default confidence ball $\tilde{B}_{M,\kappa}$ defined by (8).

Corollary 2. *Let a DDM $P(\cdot|X)$ satisfy conditions $(\tilde{A}1)$ and $(\tilde{A}2)$ with some radial rate $r(\theta_0)$, $\theta_0 \in \Theta$, and some functions φ and ψ , respectively. Let $\kappa \in (0, 1)$, the default ball $\tilde{B}_{M,\kappa}$ be defined by (8) and \hat{r}_κ be its DD-radius defined by (4). Then for any $M, \delta > 0$,*

$$P_{\theta_0}(\theta_0 \notin \tilde{B}_{M,\kappa}) \leq \frac{3\varphi(\frac{2M\delta}{5})}{2} + \frac{\psi(\delta)}{1-\kappa}, \quad P_{\theta_0}(\hat{r}_\kappa \geq Mr(\theta_0)) \leq \frac{3\varphi(\frac{M}{5})}{2\kappa} + \frac{\varphi(\frac{M}{2})}{\kappa}.$$

This corollary can be used for establishing the optimality framework (1) in the same way as Propositions 2 and 3 as we outlined in Subsection 2.4, provided the functions φ and ψ from conditions $(\tilde{A}1)$ and $(\tilde{A}2)$ are bounded uniformly over appropriate sets Θ_{cov} and Θ_{size} .

A.6 Minimality of condition (A2)

Let us demonstrate that condition (A2) is in some sense the minimal condition for providing a sufficient P_{θ_0} -coverage of the $P(\cdot|X)$ -credible ball with the sharpest rate.

Proposition 4. *For a DDM $P(\cdot|X)$ on Θ and a DD-center $\hat{\theta}$, let the ball $B(\hat{\theta}, M\hat{r}_\kappa)$ be constructed according to (5) with any $\kappa \in (0, 1)$ and $M > 0$. Further, for a $\theta_0 \in \Theta$ and a radial rate $r(\theta_0)$, denote $\psi_2(\delta) = \psi_2(\delta, \varepsilon, \theta_0) = P_{\theta_0}(d(\theta_0, \hat{\theta}) \leq \delta r(\theta_0))$, $\alpha(\delta) = \alpha(\delta, \varepsilon, \theta_0) = E_{\theta_0}[P(d(\theta, \hat{\theta}) > \delta r(\theta_0)|X)]$. Then*

$$P_{\theta_0}(\theta_0 \in B(\hat{\theta}, M\hat{r}_\kappa)) \leq \psi_2(\delta M) + \alpha(\delta)\kappa^{-1} \quad \text{for any } \delta > 0.$$

Proof. In view of the definition (5), we derive

$$\begin{aligned} & P_{\theta_0}(\theta_0 \in B(\hat{\theta}, M\hat{r}_\kappa)) \\ &= P_{\theta_0}(\theta_0 \in B(\hat{\theta}, M\hat{r}_\kappa), \hat{r}_\kappa \leq \delta r(\theta_0)) + P_{\theta_0}(\theta_0 \in B(\hat{\theta}, M\hat{r}_\kappa), \hat{r}_\kappa > \delta r(\theta_0)) \\ &\leq P_{\theta_0}(d(\hat{\theta}, \theta_0) \leq \delta Mr(\theta_0)) + P_{\theta_0}(\hat{r}_\kappa > \delta r(\theta_0)) \\ &\leq P_{\theta_0}(d(\hat{\theta}, \theta_0) \leq \delta Mr(\theta_0)) + P_{\theta_0}(P(d(\hat{\theta}, \theta) \leq \delta r(\theta_0)|X) \leq 1 - \kappa) \\ &\leq P_{\theta_0}(d(\hat{\theta}, \theta_0) \leq \delta Mr(\theta_0)) + \frac{E_{\theta_0}(P(d(\hat{\theta}, \theta) > \delta r(\theta_0)|X))}{\kappa} \\ &\leq \psi_2(\delta M) + \frac{\alpha(\delta)}{\kappa}. \end{aligned}$$

□

One should interpret this proposition as follows. First, given a DD-center $\hat{\theta}$, we determine a local radial rate $r(\theta_0)$ such that $\psi_2(\delta) \leq \bar{\alpha}(\delta)$ for all $0 < \delta \leq \delta_0$, for some “small” $\bar{\alpha}(\delta)$. This describes the sharpest rate for estimating θ_0 by $\hat{\theta}$. Next, $\alpha(\delta)$ being small for small δ means that the DDM $P(\cdot|X)$ concentrates around $\hat{\theta}$ with a faster rate than $r(\theta_0)$, which can be regarded as negation of condition (A2). The above proposition says

basically that, under negation of (A2) with the sharpest rate, the coverage probability of the credible ball $B(\hat{\theta}, M\hat{r}_\kappa)$ is bounded from above. Thus, (A2) is the minimal condition if we want to have the sharpest rate and a good coverage. This quantifies the following simple intuitive idea: if the DDM $P(\cdot|X)$ contracts in the DD-center $\hat{\theta}$ faster than $r(\theta_0)$, then the resulting radius of the credible ball $B(\hat{\theta}, M\hat{r}_\kappa)$ is going to be of a smaller order than $r(\theta_0)$. But this is going to be (over-optimistically) too small if the convergence rate of the center $\hat{\theta}$ to the truth θ_0 is not faster than $r(\theta_0)$. Then the credible ball $B(\hat{\theta}, M\hat{r}_\kappa)$ will clearly miss the truth with some probability bounded away from zero.

A.7 Inverse and direct Gaussian sequence models

Model (9) is known to be the sequence version of the *inverse signal-in-white-noise model*. This model captures many of the conceptual issues associated with nonparametric estimation, with a minimum of technical complication. Gaussian white noise models are of a canonical type of model which serves as a purified approximation to some other statistical models such as nonparametric regression model, density estimation, spectral function estimation, by virtue of the so called *equivalence principle*. The statistical inference results for the generic model (9) can be conveyed to other models, according to this equivalence principle. However, in general the problem of establishing the equivalence in a precise sense is a delicate task. Below we outline the relations with some other models.

Let \mathbb{H} , \mathbb{G} be two separable Hilbert spaces and A be a continuous operator $A : \mathbb{H} \rightarrow \mathbb{G}$. Suppose we observe

$$Y = Af + \varepsilon\xi,$$

where $\varepsilon > 0$ is the noise level, ξ is Gaussian white noise on \mathbb{G} , i.e., $\langle \xi, g \rangle \sim N(0, \|g\|^2)$ and $\text{Cov}(\langle \xi, g \rangle, \langle \xi, g' \rangle) = \langle g, g' \rangle$ for any $g, g' \in \mathbb{G}$, $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote the norm and scalar product in G . The goal is to recover $f \in \mathbb{H}$. Suppose that A^*A (A^* stands for the adjoint of A) is a compact operator so that it has a complete orthonormal system of eigenvectors $\{\phi_i, i \in \mathbb{N}\}$ in \mathbb{H} with corresponding eigenvalues $\lambda_i > 0$, i.e., $A^*A\phi_i = \lambda_i\phi_i$. Then $\{\psi_i, i \in \mathbb{N}\}$, with $\psi_i = \lambda_i^{-1/2}A\phi_i$, is an orthonormal basis in \mathbb{G} , and $A^*\psi_i = \lambda_i^{-1/2}A^*A\phi_i = \lambda_i^{1/2}\phi_i$. Now, with $\theta_i = \langle f, \phi_i \rangle$, we have $A^*Af = A^*A\sum_i \theta_i\phi_i = \sum_i \lambda_i\theta_i\phi_i$, so that $\langle Af, \psi_i \rangle = \lambda_i^{-1/2}\langle Af, A\phi_i \rangle = \lambda_i^{-1/2}\langle A^*Af, \phi_i \rangle = \lambda_i^{1/2}\theta_i$. Then the Fourier coefficient of Y with respect to $\{\psi_i, i \in \mathbb{N}\}$ are $Y_i = \langle Y, \psi_i \rangle = \langle Af, \psi_i \rangle + \varepsilon\langle \xi, \psi_i \rangle = \lambda_i^{1/2}\theta_i + \varepsilon\xi_i$, or

$$X_i = \theta_i + \sigma_i\xi_i, \quad i \in \mathbb{N},$$

where $X_i = \lambda_i^{-1/2}Y_i$, $\sigma_i = \lambda_i^{-1/2}\varepsilon$ and ξ_i 's are independent $N(0, 1)$ random variables. We thus obtained the inverse signal-in-white-noise model (9), more details can be found in [6].

For the remainder of this section, we consider the direct case $\kappa_i^2 = 1$ of model (9). This model can also be derived from the generalized linear Gaussian model as introduced in [4]: for some separable Hilbert space \mathbb{H} with scalar product $\langle \cdot, \cdot \rangle$,

$$Y^{(\varepsilon)}(x) = \langle y, x \rangle + \varepsilon W(x), \quad x \in \mathbb{H},$$

where W is a so called isonormal process; see the exact definition in [4]. Take any orthonormal basis $\{b_i, i \in \mathbb{N}\}$ in \mathbb{H} and consider $X_i = Y^{(\varepsilon)}(b_i)$, $i \in \mathbb{N}$, to reduce the above model to (9).

The following model is known as the *white noise model*. We observe a stochastic process $Y^{(\varepsilon)}(t)$, $t \in [0, 1]$, satisfying the stochastic differential equation

$$dY^{(\varepsilon)}(t) = f(t)dt + \varepsilon dW(t), \quad t \in [0, 1],$$

where $f \in \mathbb{L}_2([0, 1])$ is an unknown signal and W is a standard Brownian motion which represent the noise of intensity ε . If $\{b_i(t), i \in \mathbb{N}\}$ is an orthonormal basis in $\mathbb{L}_2([0, 1])$, then the white noise model can be translated into direct version of model (9) with observations $X_i = \int_0^1 b_i(t)dY^{(\varepsilon)}(t)$ and parameter $\theta_i = \int_0^1 b_i(t)f(t)dt$, $i \in \mathbb{N}$.

As the last related example, we mention the discrete regression model:

$$Y_i = f(x_i) + e_i, \quad i \in \mathbb{N}_n, \quad (\text{S4})$$

where e_i 's are independent $N(0, \sigma^2)$, $x_i \in [0, 1]$ are deterministic distinct points and $f(t)$ is an unknown function. Let $Y = (Y_1, \dots, Y_n)^T$, $f = (f(x_1), \dots, f(x_n))^T$, $\{b_1, \dots, b_n\}$ be an orthonormal (column) basis of \mathbb{R}^n , $W = (b_1, \dots, b_n)^T$. Denote

$$X = n^{-1/2}WY, \quad \theta = n^{-1/2}Wf, \quad \varepsilon = n^{-1/2} \quad (\text{S5})$$

to reduce (S4) again to the direct version of model (9), with the convention that $\theta = (\theta_1, \dots, \theta_n, 0, 0, \dots)$ in (9) has now zero coordinates starting from $(n+1)$ -th position. Clearly, $\|\tilde{\theta} - \theta\|^2 = n^{-1}\|\tilde{f} - f\|^2$ for $\tilde{\theta} = n^{-1/2}W\tilde{f}$.

If $x_i = i/n$, $n = 2^{J+1}$ and $f(t) \in \mathbb{L}_2([0, 1])$ in (S4), we can choose a convenient wavelet basis (of regularity $r > 0$) in $\mathbb{L}_2([0, 1])$ and apply the corresponding discrete wavelet transform W in (S5) to the original data Y . Assume that the original curve f belongs to a certain scale of Besov balls (from Besov space $B_{p,q}^s$, with $\max\{0, 1/p - 1/2\} < s < r$, $p, q \geq 1$) from $\mathbb{L}_2([0, 1])$, that include among others Hölder ($B_{\infty,\infty}^s$) and Sobolev ($B_{2,2}^s$) classes of smooth functions. Then the corresponding noiseless discrete wavelet transform $n^{1/2}\theta = Wf$ belongs to the corresponding scale of Besov balls in ℓ_2 . There is a dyadic indexing of vector $n^{1/2}\theta$, but it can be reduced to the (direct) setting of (9) by an appropriate ordering; the details are nicely explained in [4].

To give an idea how, according to the equivalence principle, the results for the model (9) can be conveyed to other (equivalent) models, let us outline a possible approach to the discrete regression model (S4):

- 1) consider the discrete regression model (S4) and assume that the unknown signal f belongs to a Besov ball $B_{p,q}^s(Q)$ with an unknown smoothness s ;
- 2) apply a discrete wavelet transform, as in (S5), to the data $Y = (Y_i, i \in \mathbb{N}_n)$ from (S4) to obtain the data X of form (9);
- 3) construct the DDM $P(\theta|X)$ (11), obtain all the results for it in terms of the data X ;
- 4) by (S5), transform the DDM $P(\theta|X)$ to the DDM $P(f|Y)$ for the signal f , now in terms of the data Y from (S4);
- 5) by equivalence of the norms for θ and f , obtain the results for the DDM $P(f|Y)$ from the results for the DDM $P(\theta|X)$.

For example the resulting DDM $P(f|Y)$ will concentrate around the true f_0 from the P_{f_0} -perspective at least with the optimal minimax rate corresponding to the smoothness s . It will take a fair piece of effort to implement this outlined approach in details, but conceptually it is a straightforward matter.

A.8 Checking conditions (10) for the mildly ill-posed case

Consider conditions (10) for the mildly ill-posed case $\kappa_i^2 = i^{2p}$. As $\sigma_i^2 = \varepsilon^2 \kappa_i^2$, these are equivalent to the same conditions for the sequence $\kappa_i^2 = i^{2p}$. In these notations, conditions (10) can be rewritten as follows: for any $\rho, \gamma > 0$, $\tau_0 > 1$, there exist some positive K_1 , $K_2 = K_2(\rho)$, $K_3 = K_3(\gamma)$, $K_4 \in (0, 1)$, $\tau > 2$ and $K_5 = K_5(\tau_0)$ such that

$$\begin{aligned} (i) \quad n^{2p+1} &\leq K_1 \sum_{i=1}^n i^{2p}, \quad (ii) \quad \sum_{i \leq \rho n} i^{2p} \leq K_2 \sum_{i=1}^n i^{2p}, \\ (iii) \quad \sum_{n=1}^{\infty} e^{-\gamma n} \left(\sum_{i=1}^n i^{2p} \right) &\leq K_3, \quad (iv) \quad \sum_{i=1}^{\lfloor m/\tau \rfloor} i^{2p} \leq (1 - K_4) \sum_{i=1}^m i^{2p}, \\ (v) \quad l \lfloor l/\tau_0 \rfloor^{2p} &\geq K_5 \sum_{i=\lfloor l/\tau_0 \rfloor + 1}^l i^{2p}, \end{aligned}$$

hold for all $n \in \mathbb{N}$, all $m \geq \tau$ and all $l \geq \tau_0$.

Let us derive the constants $K_1, K_2, K_3, K_4, \tau, K_5$ for the mildly ill-posed case $\kappa_i^2 = i^{2p}$, $p \geq 0$. First, we recall elementary relations:

$$\frac{n^{2p+1}}{2p+1} = \int_0^n x^{2p} dx \leq \sum_{i=1}^n i^{2p} \leq \int_0^{n+1} x^{2p} dx = \frac{n^{2p+1}(1 + \frac{1}{n})^{2p+1}}{2p+1}. \quad (\text{S6})$$

(i) From (S6) it follows $\frac{n^{2p+1}}{2p+1} \leq \sum_{i=1}^n i^{2p}$, so that $K_1 = 2p+1$.

(ii) In view of (S6), we have

$$\sum_{i \leq \rho n} i^{2p} \leq \frac{(\rho n + 1)^{2p+1}}{2p+1} \leq \frac{n^{2p+1}(\rho + n^{-1})^{2p+1}}{2p+1} \leq (\rho + 1)^{2p+1} \sum_{i=1}^n i^{2p},$$

so that $K_2 = (\rho + 1)^{2p+1}$.

(iii) Using (S6) and the fact that $\max_{u \geq 0} (e^{-\gamma u} u^p) = e^{-p} (p/\gamma)^p$, we evaluate

$$\begin{aligned} \sum_n e^{-\gamma n} \left(\sum_{i=1}^n i^{2p} \right) &\leq \frac{2^{2p+1}}{2p+1} \sum_n e^{-\gamma n} n^{2p+1} \\ &\leq \frac{2^{2p+1} \max_{u \geq 0} (e^{-\gamma u/2} u^{2p+1})}{2p+1} \sum_n e^{-\gamma n/2} \\ &\leq \frac{2^{2p+1} e^{-(2p+1)} (2p+1)^{2p+1}}{(2p+1)(\gamma/2)^{2p+1}} \sum_n e^{-\gamma n/2} \end{aligned}$$

$$= \frac{4^{2p+1}(2p+1)^{2p}}{(e\gamma)^{2p+1}(e^{\gamma/2}-1)},$$

that is, $K_3 = \frac{4(8p+4)^{2p}}{(e\gamma)^{2p+1}(e^{\gamma/2}-1)}$.

(iv) Denote for brevity $m_\tau = \lfloor m/\tau \rfloor$. Using (S6),

$$\begin{aligned} \sum_{i=1}^{m_\tau} i^{2p} &\leq \frac{m_\tau^{2p+1}(1 + \frac{1}{m_\tau})^{2p+1}}{2p+1} \leq \frac{m^{2p+1}(\frac{2}{\tau})^{2p+1}}{2p+1} \\ &\leq (\frac{2}{\tau})^{2p+1} \sum_{i=1}^m i^{2p} \leq \frac{1}{2} \sum_{i=1}^m i^{2p} \end{aligned}$$

if $(\frac{2}{\tau})^{2p+1} \leq \frac{1}{2}$, or $\tau \geq 2^{1+1/(2p+1)}$. Thus, we obtained $K_4 = \frac{1}{2}$ and τ can be any number satisfying $\tau \geq 2^{1+1/(2p+1)}$.

(v) Evaluate

$$\begin{aligned} \sum_{i=\lfloor l/\tau_0 \rfloor + 1}^l i^{2p} &\leq l^{2p+1} \leq l(\tau_0 \lfloor l/\tau_0 \rfloor + \tau_0)^{2p} \\ &\leq l(\tau_0 \lfloor l/\tau_0 \rfloor)^{2p} \left(1 + \frac{1}{\lfloor l/\tau_0 \rfloor}\right)^{2p} \leq l \lfloor l/\tau_0 \rfloor^{2p} (2\tau_0)^{2p}, \end{aligned}$$

so that $K_5 = (2\tau_0)^{-2p}$.

A.9 Proof of Theorem 2

Proof of Theorem 2. The proof of this theorem is essentially contained in the proof of Theorem 1. First recall that, according to (22), $\tilde{\theta} = E(\theta|X) = \sum_I X(I)P(\mathcal{I} = I|X)$, with $X(I) = \{X_i(I), i \in \mathbb{N}\} = \{X_i 1\{i \leq I\}, i \in \mathbb{N}\}$. Now, by the Fubini theorem and the Cauchy-Schwarz inequality,

$$\begin{aligned} E_{\theta_0} \|\tilde{\theta} - \theta_0\|^2 &= E_{\theta_0} \sum_i \left(\sum_I X_i(I)P(\mathcal{I} = I|X) - \theta_{0,i} \right)^2 \\ &\leq E_{\theta_0} \sum_i \sum_I (X_i(I) - \theta_{0,i})^2 P(\mathcal{I} = I|X) \\ &= E_{\theta_0} \sum_I \|X(I) - \theta_0\|^2 P(\mathcal{I} = I|X) \\ &= E_{\theta_0} \sum_I \left(\sum_{i \leq I} \sigma_i^2 \xi_i^2 + \sum_{i > I} \theta_{0,i}^2 \right) P(\mathcal{I} = I|X) \\ &\leq M^2 r^2(\theta_0) E_{\theta_0} (T_1 + T_2 + T_3), \end{aligned}$$

where T_1, T_2, T_3 are defined in (34). In the last step of the proof of Theorem 1, it is established that $E_{\theta_0}(T_1 + T_2 + T_3) \leq \frac{C_{\text{or}}}{M^2}$. The theorem follows with the constant $C_{\text{est}} = C_{\text{or}}$. \square

The local rate $r(I, \theta_0)$ defined by (17) is also the ℓ_2 -risk of the projection estimator $\hat{\theta}(I) = X(I)$: $E_{\theta_0} \|\hat{\theta}(I) - \theta_0\|^2 = r^2(I, \theta_0)$. One can regard the oracle rate (19) as the smallest possible risk over the family of (projection) estimators $\hat{\Theta}(\mathbb{N}) = \{\hat{\theta}(I), I \in \mathbb{N}\}$, namely

$$r^2(\theta_0) = r^2(I_o, \theta_0) = \inf_{I \in \mathbb{N}} E_{\theta_0} \|\hat{\theta}(I) - \theta_0\|^2 = E_{\theta_0} \|\hat{\theta}(I_o) - \theta_0\|^2.$$

Theorem 2 claims basically that the estimator $\tilde{\theta}$ given by (22) *mimics the projection oracle estimator* $\hat{\theta}(I_o)$, which is, strictly speaking, not an estimator as it depends on the true θ_0 through $I_o = I_o(\theta_0)$.

A.10 Notion of covering by a local rate

Recall that all the quantities involved depend on the information parameter ε , but we skip this dependence here. Suppose we have a family of local rates $\mathcal{R}(\mathcal{A}) = \{r(\alpha, \theta), \alpha \in \mathcal{A}\}$, e.g., in our case the family defined by (17) with $\mathcal{A} = \mathbb{N}$. Let $r(\theta) = \inf_{\alpha \in \mathcal{A}} r(\alpha, \theta)$ be the smallest local rate over $\mathcal{R}(\mathcal{A})$, called the *oracle rate*. If $r(\theta) = r(\alpha_o, \theta)$ for some $\alpha_o = \alpha_o(\theta) \in \mathcal{A}$, we call this value *oracle*.

We say that the family $\mathcal{R}(\mathcal{A})$ *covers* a scale $\Theta(\mathcal{B}) = \{\Theta_\beta, \beta \in \mathcal{B}\}$ with the corresponding family of minimax rates $\{R(\Theta_\beta), \beta \in \mathcal{B}\}$ if for any $\beta \in \mathcal{B}$ there exists an $\alpha = \alpha(\beta) \in \mathcal{A}$ such that $r(\alpha(\beta), \theta) \leq cR(\Theta_\beta)$ for all $\theta \in \Theta_\beta$ and some uniform c . Basically, this means that the family $\mathcal{R}(\mathcal{A})$ is rich enough to contain the minimax rates over the whole scale $\Theta(\mathcal{B})$. Then, for all $\beta \in \mathcal{B}$,

$$r(\theta) \leq cR(\Theta_\beta) \text{ for all } \theta \in \Theta_\beta, \quad \text{so that} \quad \sup_{\theta \in \Theta_\beta} r(\theta) \leq cR(\Theta_\beta),$$

which is the property (3). If the above property holds for some local rate $r(\theta)$ (not necessarily associated with some family of rates), we say that the *local rate* $r(\theta)$ *covers* $\Theta(\mathcal{B})$. As we already discussed in the paper, the local results with a local radial rate $r(\theta)$ imply the global minimax results for all scales which are covered by the radial rate $r(\theta)$. Therefore, in order to motivate the obtained local results, one needs to ensure this property at least for some interesting scales.

We can extend the idea of *covering* to two different families of local rates. We say that a family of local rates $\mathcal{R}_1(\mathcal{A}) = \{r_1(\alpha, \theta), \alpha \in \mathcal{A}\}$ *covers* another family of local rates $\mathcal{R}_2(\mathcal{B}) = \{r_2(\beta, \theta), \beta \in \mathcal{B}\}$ over some Θ_0 if for each $\theta \in \Theta_0$ and $\beta \in \mathcal{B}$ there exists an $\alpha = \alpha(\theta, \beta)$ such that for some uniform constant $c = c(\Theta_0, \mathcal{A}, \mathcal{B})$

$$r_1(\alpha, \theta) \leq cr_2(\beta, \theta).$$

This leads of course to the relation between the oracle rates: $r_1(\theta) \leq r_2(\theta)$ for all $\theta \in \Theta_0$. If Θ_0 contains the set of interest (e.g., $\Theta_0 = \Theta$ is the whole space), then clearly a DDM-contraction result with the oracle rate over the family $\mathcal{R}_1(\mathcal{A})$ will immediately imply the DDM-contraction result with the oracle rate over the family $\mathcal{R}_2(\mathcal{B})$.

For example, it can be easily shown that our family of local rates $\mathcal{R}(\mathbb{N}) = \{r(I, \theta), I \in \mathbb{N}\}$ defined by (17) covers the family of local radial rates $\mathcal{R}_1(\mathbb{R}_+) = \{R_{lin}(\lambda, \theta), \lambda \in \mathbb{R}_+\}$

$\Lambda_1(\mathcal{R}_+)$, where $R_{lin}^2(\lambda, \theta) = \sum_i [\sigma_i^2 \lambda_i^2 + (1 - \lambda_i)^2 \theta_i^2]$ is the risk of the linear estimator $\hat{\theta}(\lambda) = (\lambda_i X_i, i \in \mathbb{N})$ with the weights $\lambda = (\lambda_i, i \in \mathbb{N})$, and

$$\Lambda_1(\mathcal{R}_+) = \left\{ \lambda(\beta) = (\lambda_i(\beta), i \in \mathbb{N}) : \lambda_i(\beta) = \frac{i^{-(2\beta+1)}}{\sigma_i^2 + i^{-(2\beta+1)}}, \beta \in \mathcal{R}_+ \right\}.$$

This is the family of the risks of the minimax estimators over the Sobolev smoothness scale $\{\mathcal{E}_S(\beta, Q), \beta > 0\}$, where $\mathcal{E}_S(\beta, Q)$ is defined by (S12b). This is also the family of posterior convergence rates for the prior $\theta \sim \pi_\beta = \bigotimes_i N(0, i^{-(2\beta+1)})$; cf. [11] and [2] (for the direct case $\kappa_i^2 = 1$).

In fact, $\mathcal{R}(\mathbb{N})$ covers even the richer family of local rates $\mathcal{R}_2(\Lambda_{mon}) = \{R_{lin}(\lambda, \theta), \lambda \in \Lambda_{mon}\}$, where

$$\Lambda_{mon} = \left\{ \lambda = (\lambda_i, i \in \mathbb{N}) : \lambda_i \in [0, 1], \lambda_i \geq \lambda_{i+1}, i \in \mathbb{N} \right\}. \quad (\text{S7})$$

This is the family of risks of the linear estimators $\hat{\theta}(\lambda)$, with monotone weights $\lambda \in \Lambda$. Indeed, for any $\lambda \in \Lambda_{mon}$ take $N_\lambda = \max\{i : \lambda_i \geq 1/2\}$ to derive

$$\begin{aligned} R_{lin}^2(\lambda, \theta) &= \sum_i [\sigma_i^2 \lambda_i^2 + (1 - \lambda_i)^2 \theta_i^2] \geq \sum_{i \leq N_\lambda} \frac{\sigma_i^2}{4} + \sum_{i > N_\lambda + 1} \frac{\theta_i^2}{4} \\ &= \frac{r^2(N_\lambda, \theta)}{4} \geq \frac{r^2(I_o, \theta)}{4}. \end{aligned}$$

Clearly, $\mathcal{R}_1(\mathbb{R}_+) \subset \mathcal{R}_2(\Lambda)$. Besides, $\mathcal{R}_2(\Lambda_{mon})$ contains also the family of risks of the minimax Pinskers estimators (which are asymptotically minimax over Sobolev ellipsoids up to the constant) and the family of risks of the (minimax) Tikhonov regularization estimators, which correspond to spline estimators in the problem of curve estimation.

A.11 Proof of (21)

Recall the definitions (20) of ellipsoid $\mathcal{E}(a)$ and hyperrectangle $\mathcal{H}(a)$. First consider the hyperrectangles $\mathcal{H}(a)$. It follows from [7] that

$$\begin{aligned} R^2(\mathcal{H}(a)) &= \inf_{\hat{\theta}} \sup_{\theta \in \mathcal{H}(a)} \mathbb{E}_\theta \|\hat{\theta} - \theta\|^2 \geq \frac{4}{5} \inf_b \sup_{\theta \in \mathcal{H}(a)} \mathbb{E}_\theta \|\tilde{\theta}(b) - \theta\|^2 \\ &= \frac{4}{5} \sum_i \frac{a_i^2 \sigma_i^2}{a_i^2 + \sigma_i^2}, \end{aligned}$$

where $\tilde{\theta}(b) = (\tilde{\theta}_i(b), i \in \mathbb{N})$, $b = (b_i \in \mathbb{R}, i \in \mathbb{N})$, is the class of linear estimators $\tilde{\theta}_i(b) = b_i X_i$. Take $N_a = \max\{i : \sigma_i^2 \leq a_i^2\}$, then for any $\theta_0 \in \mathcal{H}(a)$ (for some *unknown* a) we have

$$\begin{aligned} \sum_i \frac{a_i^2 \sigma_i^2}{a_i^2 + \sigma_i^2} &\geq \sum_{i \leq N_a} \frac{\sigma_i^2}{2} + \sum_{i > N_a} \frac{a_i^2}{2} \geq \frac{1}{2} \inf_I \left\{ \sum_{i \leq N_a} \sigma_i^2 + \sum_{i > I} a_i^2 \right\} \\ &\geq \frac{1}{2} \inf_I \left\{ \sum_{i \leq I} \sigma_i^2 + \sum_{i > I} \theta_{0,i}^2 \right\} = \frac{r^2(\theta_0)}{2}. \end{aligned}$$

Combining the last two relation yields the second bound in (21).

Now suppose $\theta_0 \in \mathcal{E}(a)$ for some *unknown* a . From [3] it follows that for any $\theta_0 \in \mathcal{E}(a)$ and some $\dot{\lambda} = (\dot{\lambda}_i, i \in \mathbb{N}) \in \Lambda_{mon}$ (Λ_{mon} is defined by (S7)):

$$\begin{aligned}
R^2(\mathcal{E}(a)) &\geq \inf_{\lambda} \sup_{\theta \in \mathcal{E}(\pi a)} R_{lin}(\lambda, \theta) \\
&\geq \pi^{-2} \inf_{\lambda} \sup_{\theta \in \mathcal{E}(a)} R_{lin}(\lambda, \theta) = \pi^{-2} \sup_{\theta \in \mathcal{E}(a)} R_{lin}(\dot{\lambda}, \theta) \\
&= \pi^{-2} \sup_{\theta \in \mathcal{E}(a)} \sum_i [\dot{\lambda}_i^2 \sigma_i^2 + (1 - \dot{\lambda}_i)^2 \theta_i^2] \\
&\geq \pi^{-2} \left[\sum_{i: \dot{\lambda}_i \geq 1/2} \frac{\sigma_i^2}{4} + \sup_{\theta \in \mathcal{E}(a)} \sum_{i: \dot{\lambda}_i < 1/2} \frac{\theta_i^2}{4} \right] \\
&\geq (2\pi)^{-2} \left(\sum_{i \leq N_{\dot{\lambda}}} \sigma_i^2 + a_{N_{\dot{\lambda}}+1}^2 \right) \geq (2\pi)^{-2} \inf_I \left\{ \sum_{i \leq I} \sigma_i^2 + a_{I+1}^2 \right\} \\
&\geq (2\pi)^{-2} \inf_I \left\{ \sum_{i \leq I} \sigma_i^2 + \sum_{i > I} \theta_{0,i}^2 \right\} = (2\pi)^{-2} r^2(\theta_0),
\end{aligned}$$

which leads to the first bound in (21).

The exact form of weights $\dot{\lambda} \in \Lambda_{mon}$ is not important, but we just remind here that these are the so called Pinsker optimal weights (cf. [9]): $\dot{\lambda}_i = (1 - \dot{\mu}/a_i)_+$, where $x_+ = x \vee 0$ and $\dot{\mu} = \dot{\mu}(\sigma, a)$ is the unique solution of the equation

$$\sum_i \sigma_i^2 (1 - \dot{\mu}/a_i)_+ / (a_i \dot{\mu}) = 1.$$

The constant $(2\pi)^{-2}$ is actually too conservative. For example, for the direct case $\kappa_i^2 = 1$, it follows from [7] (see also Proposition 3 in [4]) that

$$\begin{aligned}
R^2(\mathcal{E}(a)) &= \inf_{\hat{\theta}} \sup_{\theta \in \mathcal{E}(a)} \mathbb{E}_{\theta} \|\hat{\theta} - \theta\|^2 \geq (4.44)^{-1} \inf_I \{I\epsilon^2 + a_{I+1}^2\} \\
&\geq (4.44)^{-1} \inf_I \left\{ I\epsilon^2 + \sum_{i > I} \theta_{0,i}^2 \right\} \geq (4.44)^{-1} r^2(\theta_0).
\end{aligned}$$

Since $R^2(\mathcal{E}(a)) \leq \inf_I \sup_{\theta \in \mathcal{E}(a)} \mathbb{E}_{\theta} \|X(I) - \theta\|^2 = \inf_I \left\{ \sum_{i \leq I} \sigma_i^2 + a_{I+1}^2 \right\}$ and $R^2(\mathcal{H}(a)) \leq \inf_I \sup_{\theta \in \mathcal{H}(a)} \mathbb{E}_{\theta} \|X(I) - \theta\|^2 = \inf_I \left\{ \sum_{i \leq I} \sigma_i^2 + \sum_{i > I} a_i^2 \right\}$, we conclude that

$$R^2(\mathcal{E}(a)) \asymp \inf_I \left\{ \sum_{i \leq I} \sigma_i^2 + a_{I+1}^2 \right\}, \quad R^2(\mathcal{H}(a)) \asymp \inf_I \left\{ \sum_{i \leq I} \sigma_i^2 + \sum_{i > I} a_i^2 \right\}. \quad (\text{S8})$$

A.12 Other choices for DDM, over-shrinkage effect

Notice that we do observe the Bayesian tradition as our DDM $P(\cdot|X)$ defined by (11) results from certain empirical Bayes posterior. However, in principle we can manipulate with different ingredients in constructing DDMs: different choices for $P_I(\cdot|X)$ and $P(\mathcal{I} = I|X)$ in (11) are possible, not necessarily coming from the (same) Bayesian approach.

In (14), we could take $\tau_i^2(I) = K_1\sigma_i^2 1\{i \leq I\} + K_2\sigma_i^2 1\{i > I\}$ for some $0 \leq K_2 < K_1$ (the choice in (14) is a particular case with $0 = K_2 < K_1 = K$), to possibly improve constants in the main results by choosing appropriate K_2 , but this would further complicate the expressions without gaining anything conceptually.

Empirical Bayes posterior with respect to I One more choice for DDM within Bayesian tradition is the empirical Bayes posterior $\hat{P}(\cdot|X)$ with respect to I introduced by (16) in Remark 3. We remind its definition:

$$\hat{P}(\cdot|X) = P_{\hat{I}}(\cdot|X), \quad \text{with} \quad \hat{I} = \min \left\{ \operatorname{argmax}_{I \in \mathbb{N}} P(\mathcal{I} = I|X) \right\},$$

where $P_I(\cdot|X)$ and $P(\mathcal{I} = I|X)$ are defined by respectively (12) and (13). The argmax gives a subset of \mathbb{N} in general, \hat{I} is the smallest element in this set.

Let us demonstrate that the DDM $\hat{P}(\cdot|X)$ has exactly the same properties as the DDM $P(\cdot|X)$ defined by (11). By the definition of \hat{I} , we derive that, for any $I, I_0 \in \mathbb{N}$ and any $h \in [0, 1]$,

$$P_{\theta_0}(\hat{I} = I) \leq P_{\theta_0} \left(\frac{P(\mathcal{I} = I|X)}{P(\mathcal{I} = I_0|X)} \geq 1 \right) \leq E_{\theta_0} \left[\frac{P(\mathcal{I} = I|X)}{P(\mathcal{I} = I_0|X)} \right]^h,$$

which yields the analogue of (27). From this point on, the proof of the properties of the DDM $\hat{P}(\cdot|X)$ proceeds exactly in the same way as the proof for the DDM $P(\cdot|X)$ defined by (11), with the only difference that everywhere (in the claims and in the proofs), $1\{\hat{I} = I\}$ is substituted instead of $P(\mathcal{I} = I|X)$ and $P_{\theta_0}(\hat{I} = I)$ is substituted instead of $E_{\theta_0}P(\mathcal{I} = I|X)$.

An interesting connection of this DDM to penalized estimators is discussed in Subsection A.13.

Other choices for $P_I(\cdot|X)$ For example, if we only were interested in the upper bound result (Theorem 1) for the resulting $P(\cdot|X)$ (11), instead of the DDM (12) we could use $P_I(\cdot|X) = \bigotimes_i N(X_i 1\{i \leq I\}, \sigma_i^2(I))$ with any variances $\sigma_i^2(I)$ such that $\sum_i \sigma_i^2(I) \leq C \sum_{i \leq I} \sigma_i^2$ for some $C > 0$. Even the degenerate DDM $P_I(\cdot|X)$ with $\sigma_i^2(I) = 0$ (or $L = 0$ in (12)) would lead to the oracle DDM-contraction rate. On the other hand, this choice would however make the lower bound result (Theorem 3) impossible to hold. In fact, non-normal distributions in the construction of the DDMs $P_I(\cdot|X)$ are also possible as we only use the Markov inequality when dealing with $P_I(\cdot|X)$, just the right choice of the first two moments would be sufficient for the upper bound result. However, when proving the lower bound result, Theorem 3, we need to deal with a small ball probability, which is a relatively well studied problem for the Gaussian distribution. For a non-normal case, one would first have to derive small ball probability results.

Other choices for $P(\mathcal{I} = I|X)$ Instead of the mixing DDM $P(\mathcal{I} = I|X)$ (13) in (11), the main results would also hold for the following DDM:

$$\Pi'(\mathcal{I} = I|X) = \frac{\lambda_I \bigotimes_i \varphi(X_i, 0, \sigma_i^2 + \tau_i^2(I))}{\sum_J \lambda_J \bigotimes_i \varphi(X_i, 0, \sigma_i^2 + \tau_i^2(J))}, \quad I \in \mathbb{N}, \quad (\text{S9})$$

with $\tau_i^2(I)$ and λ_I defined by (14). The DDM $\Pi'(\mathcal{I} = I|X)$ defined by (S9) is nothing else but the posterior probability of \mathcal{I} with respect to the prior (15) with $\mu_i(I) = 0$ for all $i, I \in \mathbb{N}$; we denote this prior by Π' . The right hand side of (S9) means the P_{θ_0} almost sure limit

$$\begin{aligned}\Pi'(\mathcal{I} = I|X) &= \lim_{m \rightarrow \infty} \Pi'(\mathcal{I} = I|X_1, \dots, X_m) \\ &= \lim_{m \rightarrow \infty} \frac{\lambda_I \bigotimes_{i=1}^m \varphi(X_i, 0, \sigma_i^2 + \tau_i^2(I))}{\sum_J \lambda_J \bigotimes_{i=1}^m \varphi(X_i, 0, \sigma_i^2 + \tau_i^2(J))},\end{aligned}$$

which exists by the martingale convergence theorem.

“Over-shrinkage” effect of (mixtures of) normal priors Although the prior Π' (the prior defined by (15) with $\mu_i(I) = 0$ for all $i, I \in \mathbb{N}$) leads to the “correct” posterior (S9) on I (in the sense that it can be used instead of the DDM (13) in (11)), it yields the “over-shrunk” resulting posterior on θ . Indeed,

$$\Pi'_I(\cdot|X) = \Pi'(\cdot|X, \mathcal{I} = I) = \bigotimes_i N(LX_i(I), L\sigma_i^2 1\{i \leq I\}), \quad (\text{S10})$$

with $L = \frac{K}{K+1} < 1$, so that the actual resulting posterior of θ

$$\Pi'(\cdot|X) = \sum_I \Pi'_I(\cdot|X) \Pi'(\mathcal{I} = I|X) \quad (\text{S11})$$

contracts, from the P_{θ_0} -perspective, to $L\theta_0$ and not to θ_0 . This has to do with the *shrinkage effect* of (mixtures of) normal priors towards the prior mean, which is inherent to the normal-normal model. This has already been observed in [8], and discussed at length by [1] and [5]. The approaches in the first and third papers are based on (mixtures of) heavy-tailed priors instead of normal. A related approach is to add one more level of hierarchy in (15) by putting a heavy-tailed prior on variances $\tau_i^2(I)$. This will of course again destroy the normal conjugate structure of the prior, whereas normal/mixture-of-normals model has an advantageous feature that all the quantities involved can be explicitly computed and controlled.

Basically, a “correct” DDM $\Pi'_I(\cdot|X)$ in the expression (S11) should be of the form $\Pi'_I(\cdot|X) = \bigotimes_i N(X_i(I), L\sigma_i^2 1\{i \leq I\})$ for any $L > 0$. Within the DDM methodology, one can, in principle, adjust the posterior (S10) by blowing it up (by the factor L^{-1}) or by shifting it (by the factor $(1-L)X(I)$), or one can simply use the DDM (12) instead of (S10). However, such manipulations with posteriors are not done by the committed Bayesians. If one insists on normal mixture prior and wants to get a correct posterior (S10), the only way to achieve this is to take the prior variances $\tau_i^2 \gg \sigma_i^2$, in the asymptotic sense as $\varepsilon \rightarrow 0$, so that $L \approx 1$. However, this makes the whole consideration necessarily asymptotic. A more important issue with this approach is that we were unable to derive good concentration properties for the posterior $\Pi'(\mathcal{I} = I|X)$ in this case.

Thus, for a Bayesian who would like to use normal/mixture-of-normals model, there is a following dilemma: if the prior variances τ_i^2 are of order σ_i^2 , we obtain a “correct” $\Pi'(\mathcal{I} = I|X)$, but over-shrunk (towards prior mean) $\Pi'_I(\cdot|X)$ ’s; on the other hand, if $\tau_i^2 \gg$

σ_i^2 , $\Pi'_I(\cdot|X)$'s are then “correct”, but $\Pi'(\mathcal{I} = I|X)$ does not possess good concentration properties (at least we were unable to establish this).

The empirical Bayes approach resolves this issue, also within the Bayesian paradigm, as we demonstrated in the paper. The idea is to treat the prior means as parameters chosen by the empirical Bayes procedure, which removes the over-shrinkage effect.

A.13 Connection to penalized estimators

In view of Remark 3, Theorem 4 also holds for the DDM $P_{\hat{I}}(\cdot|X)$ defined by (16), instead of the DDM $P(\cdot|X)$ given by (11). If we take the DDM-expectation with respect to the DDM $P_{\hat{I}}(\cdot|X)$ (like we did in (22) for the DDM $P(\cdot|X)$), we obtain the estimator

$$\hat{\theta} = X(\hat{I}) = (X_i 1\{i \leq \hat{I}\}, i \in \mathbb{N}).$$

In the direct case $\kappa_i^2 = 1$, some basic computations reveal that \hat{I} is the minimizer of

$$\text{crit}(I) = -\|X(I)\|^2 + (\log(K+1) + 2\alpha)\varepsilon^2 I,$$

so that $\hat{\theta} = X(\hat{I})$ turns out to be the so called *penalized projection estimator* with the penalty constant $P(K, \alpha) = \log(K+1) + 2\alpha$, studied by [4].

Interestingly, the conditions $K \geq 1.87$ and $a(K) > \alpha > 0$, coming from Theorems 1 and 3, lead to the following range for the penalty constant: $P(K, \alpha) \in [1.05, 1.2]$. Although this is probably not the most precise range, the fact itself (that $P(K, \alpha) \in [1.05, 1.2]$) reconfirms, from a different perspective, the conclusion of [4] that the penalty constant should certainly be bigger than 1, but not too large.

A.14 Relation to the results of [11]

Here we demonstrate that our local results for the DDM $P(\cdot|X)$ defined by (11) imply, among others, the non-asymptotic versions of the global minimax results obtained in the intriguing paper by Szabó, van der Vaart and van Zanten [11]. In our notations, the observations in [11] are $X' = (X'_i, i \in \mathbb{N}) \sim P_{\theta_0} = P_{\theta_0}^{(n)} = \bigotimes_i N(\theta_{0,i} \kappa_i^{-1}, n^{-1})$, which is effectively the same model as (9) with $X'_i = \kappa_i^{-1} X_i$ and $n^{-1/2} = \varepsilon$. A family of priors on θ is considered in [11]: $\Pi_\alpha = \bigotimes_i N(0, i^{-(2\alpha+1)})$, $\alpha \in [0, A]$, leading to the posteriors $\Pi_\alpha(\cdot|X')$ and the marginal distributions $X' \sim \Pi_{\alpha, X'}$ with the (marginal) likelihood $\ell_n(\alpha) = \ell_n(\alpha, X')$. The proposed DDM is $\Pi_{\hat{\alpha}_n}(\cdot|X')$ with $\hat{\alpha}_n = \arg\max_{\alpha \in [0, A]} \ell_n(\alpha)$, which is the empirical Bayes posterior with respect to the parameter α . The DDM $\Pi_{\hat{\alpha}_n}(\cdot|X')$ is then used to construct a DDM-credible ball whose coverage and size properties were studied.

The main results in [11] are the asymptotic (as $n \rightarrow \infty$ or, in our notations, as $\varepsilon \rightarrow 0$) versions of the minimax framework (2), for $\Theta'_{cov} = \Theta_{pt}$ and the following four scales: Sobolev hyperrectangles \mathcal{H}_S , Sobolev ellipsoids \mathcal{E}_S and the two *supersmooth* scales, analytic ellipsoids \mathcal{E}_A and parametric hyperrectangles \mathcal{H}_P (the notations in [11] are $\Theta^\beta(Q)$, $S^\beta(Q)$, $S^{\infty, c, d}(Q)$ and $C^{00}(N_0, Q)$, respectively). Precisely, let $Q, \beta, c, d > 0$, $N_0 \in \mathbb{N}$, and $\mathcal{E}(a), \mathcal{H}(a)$ be defined by (20). Then

$$\mathcal{H}_S = \mathcal{H}_S(\beta, Q) = \mathcal{H}(a) \quad \text{with} \quad a_i^2 = Q i^{-(2\beta+1)}, \quad (\text{S12a})$$

$$\mathcal{E}_S = \mathcal{E}_S(\beta, Q) = \mathcal{E}(a) \text{ with } a_i^2 = Qi^{-2\beta}, \quad (\text{S12b})$$

$$\mathcal{E}_A = \mathcal{E}_A(c, d, Q) = \mathcal{E}(a) \text{ with } a_i^2 = Qe^{-ci^d}, \quad (\text{S12c})$$

$$\mathcal{H}_P = \mathcal{H}_P(N_0, Q) = \mathcal{H}(a) \text{ with } a_i^2 = Q1\{i \leq N_0\}. \quad (\text{S12d})$$

By using (S8), it is easy to compute the corresponding minimax rates over these scales, under the asymptotic regime $\varepsilon \rightarrow 0$ (or, $n \rightarrow \infty$):

$$\begin{aligned} R^2(\mathcal{E}_S) &\asymp \varepsilon^{4\beta/(2\beta+2p+1)} = n^{-2\beta/(2\beta+2p+1)}, \\ R^2(\mathcal{H}_S) &\asymp \varepsilon^{4\beta/(2\beta+2p+1)} = n^{-2\beta/(2\beta+2p+1)}, \\ R^2(\mathcal{E}_A) &\asymp \varepsilon^2(\log \varepsilon^{-1})^{(2p+1)/d} = \frac{(\log n)^{(2p+1)/d}}{n}, \\ R^2(\mathcal{H}_P) &\asymp \varepsilon^2 = n^{-1}. \end{aligned}$$

Notice that the parametric class \mathcal{H}_P automatically satisfies EBR.

The DDM $\Pi_{\hat{\alpha}_n}(\cdot|Y)$ is well suited to model Sobolev-type scales: the optimal (minimax) radial rates are obtained in the size relation of (2) for Sobolev hyperrectangles \mathcal{H}_S and ellipsoids \mathcal{E}_S , but only suboptimal rates for the two supersmooth scales \mathcal{E}_A and \mathcal{H}_P :

$$\begin{aligned} \frac{(\log n)^{(p+1/2)\sqrt{\log n}}}{n} &\gg R^2(\mathcal{E}_A) = \frac{(\log n)^{(2p+1)/d}}{n}, \\ \frac{e^{(3p+3/2)\sqrt{\log N_0}\sqrt{\log n}}}{n} &\gg R^2(\mathcal{H}_P) = \frac{1}{n}. \end{aligned}$$

For the DDM $P(\cdot|X)$ defined by (11), Theorem 4 implies, in view of $\Theta_{pt} \subseteq \Theta_{eb}$ and (21), the non-asymptotic minimax results (2) for all ellipsoids $\mathcal{E}(a)$ and hyperrectangles $\mathcal{H}(a)$ defined by (20), for *all unknown* (non-increasing) a . Now note that, according to (S12a)–(S12d), the four above mentioned scales from [11] are particular examples of ellipsoids $\mathcal{E}(a)$ and hyperrectangles $\mathcal{H}(a)$, with specific choices of sequence a . Hence, the minimax results (2) for all the four scales (including the two supersmooth scales \mathcal{E}_A and \mathcal{H}_P) follow for the DDM (11). Asymptotic version can readily be derived from the non-asymptotic ones. Recall that the scope of the DDM $P(\cdot|X)$ extends further than the above mentioned four scales, even beyond general families of ellipsoids and hyperrectangles. The local results of Theorem 4 deliver the minimax results of type (2) for all scales for which (3) holds; for example, also for the scales of tail classes and ℓ_p -bodies.

References

- [1] BABENKO, A. and BELITSER, E. (2009). Oracle projection convergence rate of posterior. *Math. Meth. Statist.* 19, 219–245.
- [2] BELITSER, E. and GHOSAL, S. (2003). Adaptive Bayesian inference on the mean of an infinite dimensional normal distribution. *Ann. Statist.* 31, 536–559.
- [3] BELITSER, E. and LEVIT, B. (1995). On minimax filtering over ellipsoids. *Math. Meth. Statist.* 3, 259–273.

- [4] BIRGÉ, L. and MASSART, P. (2001). Gaussian model selection. *J. Eur. Math. Soc.* 3 203–268.
- [5] CASTILLO, I. and VAN DER VAART, A.W. (2012). Needles and straw in a haystack: posterior concentration for possibly sparse sequences *Ann. Statist.* 40, 2069–2101.
- [6] CAVALIER, L. (2008). Nonparametric statistical inverse problems. *Inverse Problems* 24, 1–19.
- [7] DONOHO, D.L., LIU, R.C. and MACGIBBON, B., (1990). Minimax risk over hyperrectangles, and implications. *Ann. Statist.* 18, 1416–1437.
- [8] JOHNSTONE, I.M. and SILVERMAN, B.W. (2004). Needles and straw in haystacks: empirical Bayes estimates of possibly sparse sequences. *Ann. Statist.* 32, 1594–1649.
- [9] PINSKER, M.S. (1980). Optimal filtration of square-integrable signals in Gaussian noise. *Problems Inform. Transmission* 16, 120–133.
- [10] ROBINS, J. and VAN DER VAART, A.W. (2006). Adaptive nonparametric confidence sets. *Ann. Statist.* 34, 229–253.
- [11] SZABÓ, B.T., VAN DER VAART, A.W. and VAN ZANTEN, J.H. (2015). Frequentist coverage of adaptive nonparametric Bayesian credible sets. *Ann. Statist.* 43, 1391–1428.